THE

Mathematics Student

A Quarterly Dedicated to the Service of Students and Teachers of Mathematics in India.

VOL. VIII.—1940

Edited by

A. NARASINGA RAO, M.A., L.T., D.Sc., F.A.Sc.

Annamalai University, Annamalainagar, South India.

With the co-operation of

N. DURAIRAJAN, M.A., B.E.
V. GANAPATI IYER, M.A., D.Sc.
A. A. KRISHNASWAMI AYYANGAR, M.A., L.T.
B. RAMAMURTI, M.A., D.Sc.
S. SIVASANKARANARAYANA PILLAI, D.Sc.
H. SUBRAMANI IYER, M.A., Ph.D.

and others.

Published by
The Indian Mathematical Society

ST. JOSEPH'S INDUSTRIAL SCHOOL PRESS, TRICHY.
1940

List of exchanges with the Publications of the Indian Mathematical Society

India

1. Proc. of the National Institute of Sciences of India.
2. Trans. of the National Institute of Sciences of India.
3. Proc. of the Indian Academy of Sciences.
4. Current Science.
5. Bulletin of the Calcutta Math. Society.

Japan

1. Collected Papers from the Faculty of Science, Osaka Imperial University.
2. Japanese Journal of Mathematics.
3. Journal of the Faculty of Science, Hokkaido Imperial University.
4. Proc. of the Physico Mathematical Society of Japan.
5. Tohoku Mathematical Journal.

England

1. Monthly Notices of Royal Astronomical Society.
2. Proc. of Cambridge Phil. Society.
3. Proc. of the Edinburgh Mathematical Society.
4. Journal of the London Mat matical Society.
5. Mathematical Gazette.
6. Math. Notes of the Edinbur Math. Society.
7. Science Progress.
8. Quarterly Journal of Mat *matics*

Sweden

1. Acta Mathematica.
2. Arkiv fur Mathematik Ast nomie ok Physik.
3. Lund University Journal.

Holland

Nieuw Archief Voor W kunde.

Belgium

Sphinx.

France

1. Annales de l'Institut He Poincare.
2. Annales Scientifiques de Ecole Normale Superieu
3. Bulletin des Sciences Mat matiques.
4. Comptes Rendus de l'Aca mie des Sciences.

THE

Mathematics Student

A Quarterly Dedicated to the Service of Students and Teachers of Mathematics in India

VOL VIII 1940

Edited by

A. NARASINGA RAO [illegible]

[illegible]

N. DURAIRAJAN, M.A. [illegible]
V. GANAPATI IYER [illegible]
A. A. KRISHNASWAMI AYYANGAR [illegible]

Published by
The Indian Mathematical Society

St. Joseph's Industrial School Press, Trichy
1940

CONTENTS

PAGE

Augustine, K. O. (Konnully, A): A note on Cauchy's theorem 82— 83

———————: Pedal conic ... ... 180—181

Bagchi, N.: An alternate method of finding out the square root of a number ... ... 177—180

Balachandran, K.: Power series for circular functions ... 174—177

Chidambaram, S.: On a particular inscribed rectangle of a conic ... . 143—147

Chowla S. and Auluck F. C.: An approximation with exp. x 75— 77

———————: On Weirstrass approximation theorem 78— 79

Gupta, H.: A problem in combinations ... ... 131—132

Gunjukar K. R. and Oza. H. P.: Joule's minimum dissipation theorem 40

Kesava Menon, P.: A generalisation of the circular and hyperbolic functions ... ... 112—117

——————— A theorem on Congruence ... ... 156—158

Krishnaswamy Ayyangar, A. A: Lecture notes ... 83— 88

———————: Rene Descartes ... ... 101—108

———————: The role of unit partial quotients in some continued fractions ... ... 159—166

Kosambi, D. D.: A note on frequency distribution in series... 151—155

Levi, F. W.: A problem on rigid notion ... ... 1— 10

Madhava Rao, B. S.: Generalised geometry and physical theories ... ... 11— 34

Miller, G. A.: Enumeration of finite groups ... ... 109—111

Narasimhamurty, V.: On demlo numbers ... ... 34— 37

Narasinga Rao, A.: Studies in circle geometry ... 53— 72

———————: A fallacy in deducing the attraction of a plane from that of a spherical shell ... 182—183

Neville, E. H.: A proof of an elementary theorem on complex numbers 173

Pillai, S. S.: On the converse of Fermat's Theorem ... 132—133

Rajagopal, C. T.: On Abel's divergence test for series of positive terms ... ... 118—123

Rangaswami, K.: A note on the harmonic envelope ... 37— 39

———————: Remarks on Mr. Augustine's note ... 181—182

Sambasiva Rao, K.: Lecture Notes 89

Sastry, B. S.: The limiting points of the set of the upper and lower sums in Riemann integration ... 148—150
Sastry, S.: On the sum of $\sum x^{12n}/(12n)!$ ... ... 133—134
Sitaraman, B.: On correlation constants in mingled records 73— 75
Sperling, Maurice.: On the number π ... ... 41— 43
Srinivasan, G.: The infinitesimal flow in the inversive plane ... ... 167—170
Srinivasiengar, C. N.: The line of umbilics ... ... 171—172
Srinivasa Sastry, H.: On the function defined by the relation $\phi_r(x) = x\phi_{r-1}(x)$ ... ... 127—130
Vaidyanatha Sastry, M.: Heliacal rising of the moon ... 80— 81
Announcements and News 48, 95, 140
Solutions to Questions ... 49—51, 96—99, 134—139, 184—186
Questions for Solution ... ...51—52, 100, 141—142
Gleanings 10, 43, 52, 130, 142, 155, 158, 166, 170
Books received for Review 47
Reviews:
  Weyrich: *Die Zylinderfunktionen and ihre Anwendungen* 44
  Vasilesco, F: *La notion de capacite* ... ... 44— 46
  ————————: *La notion de point irregulier*... ... 44— 46
  Mandelbrojt: *La regularisation des Fonctions* ... 47
  Colloque consacre a la Theorie des Probabilites: Actualites Scientifiques Nos. 734, 735, 736, 737, 738 and 739: 90— 95

Author Index of Questions and Solutions

(The numbers refer to pages)

Auluck, J. C.
Solutions 185, 186.
Balachandran, K.:
Question 141.
Solution 185.
Court, N. A.:
Question 100.
Hariharan, P.
Solutions 185^{2}, 186.
Iyengar, K. S. K.:
Question 51, 135.
Solution 136.
Kaprekar, D. R.
Question 141.
Kesava Menon, P.
Questions 100^{2}, 141, 185.
Krishnamoorthi, C. R.
Solution 185.
Krishnaswamy Ayyangar, A. A.
Questions 97, 100, 142.
Krishnaswamy, M. N.
Question 52.
Manickavasagam, P. S. J.
Solution 186.
Moessner, A.
Question 97.
Narasimhamurthy, N. K.
Solution 49.
***Narasimhamurthy*, Y. K.**
Solution 135.
Palkar, A. E.
Questions, 100, 185.
Radhakrishna Rao, C.
Solution 186.
Raghavachari, T. K.
Solution 50, 135.
Ramanathan, K. G.
Question
Ramanujam, R.
Question 96.
Solution 96.
Rathie, C. B.
Solution 98.
Regan, F.
Question 50.
Sen, D. K.
Solution 134.
Srinivasan, G.
Solutions 185, 186.
Subba Rao, K.
Solution 135.
Sriranganarayana, M. A.
Question 141.
Thebault, V.
Questions, 49^{2}, 51, 134, 184.
Solution 49^{2}, 135.
Venkatanarasimham, S.
Solutions 185^{2}.
Venkataraman.
Solution 185.
Venkatasubbiah, S.
Solution 184.
Weiland, A.
Solution 97.

Index of Questions Solved in Volume VIII

Question	Page	Question	Page
1597	96	1713	50
1668	97	1775	97
1694	49	1785	185
1700	134	1786	135
1708	49	1788	185
1791	185		

ERRATA FOR VOLUME VIII

Page	*Line*	*For*	*Read*
108	4B†	Brittannica	Britannica
114	19	$(n+2r)!$	$(2n+r)!$
128	2B	relation (4)	relation (4′)
128	8B	in (3′)	in (4′)
129	2	with (4)	with (4′)
129	11	by (4)	by (4″)
150	2B	ε	ε'
152	8	$1=np$	$=np$
156	14	$n^{\phi(m)}$	n_{ϕ}
156	15	$r\cdot n_1\, n_2 \cdots n^{\phi}$	$r\cdot n_1\, n_2 \cdots n_{\phi}$
157	10	after "(mod m')"	add "...(4)"
157	20	$p_s{}^{t} + \cdots)$	$p_s{}^{t}) + \cdots$
158	9	$\lambda m^{\phi_r}\, r$	$\lambda m \phi_r\, r$
159	8B	$\frac{1}{a_v} + \cdots$	$\frac{1}{a_v + \cdots}$
160		Delete the contents of line 3 and insert instead $\underset{*}{a_0} + \frac{1}{a_1+}\,\frac{1}{a_2+}\cdots\frac{1}{a_r+}\,\frac{1}{1+}\,\frac{1}{1+}\cdots\frac{1}{1+}\,\frac{1}{a_{r+n+1}} + \frac{1}{\underset{*}{a_p}}$	
160	7	Add at the end of the line "except unity"	
161	2B	(6)	(3)
162	2	(1)	(3·1)
165	17 and 19	$\xi_{p+2'+2}$	ξ_{p+2t+2}
166	6	ber	über
,,	22	No. 2	part X
,,	8B	Charles	Chasles
170		Turn figure so that X′OX is parallel to the shorter edges of the page	
175	8B	in (9)	in (8)
176	equation (14)	$E_2\, x$	$E_2\, x^2$
,,	9B	n^{th}	k^{th}
,,	2B	$E_2=1,\ E_4=5,\ E_6=61$	$E_2=\frac{1}{2!},\ E_4=\frac{5}{4!}\ E_6=\frac{61}{6!}$
,,	1B	$x^2+5x^4+61x^6$	$\frac{x^2}{2!}+\frac{5x^4}{4!}+\frac{61x^6}{6!}$

---

† Read "Fourth line from the bottom of the page.

THE MATHEMATICS STUDENT

Volume VIII] MARCH 1940 [Number 1

A PROBLEM ON RIGID MOTION

BY

F. W. LEVI, *Calcutta*

It is a fundamental fact of Analysis that every infinite set can be (1, 1)-represented by proper subsets of itself, whereas the same is impossible for finite sets.* If one considers rigid motions only in place of general (1, 1)-representations, an analogous proposition can be stated for linear point sets. Of course a bounded linear point set cannot be transformed by any rigid motion into a proper subset of itself, whereas certain unbounded sets can be as they contain proper subsets congruent to the original set. For example, the half rays $x>0$ and $x>1$ are congruent, and the rigid motion $x'=x+1$ transforms the first one into the second.

One may expect that the same proposition holds for point-sets of any dimension as the measure (area, volume etc.) of a bounded point-set is not altered by rigid motion, but in general point-sets have proper subsets of the same measure (the difference being not of positive measure) and it may be that these subsets are congruent to the original set.

The two-dimensional bounded point-sets which by a suitable rigid motion can be transformed into a proper subset, as well as these rigid motions themselves will be considered in this paper.

§ 1. *A characteristic example.*

Represent the points of the plane by complex numbers in the usual way, and consider the set

$$e^{2m\alpha\pi i} \qquad \dots \quad (1)$$

for $m=1, 2, \cdot\cdot\cdot\cdot$, α being any particular *irrational* number.

These points are all situated on the unit-circle, and different values of m correspond to different points. The rotation about 0 through the angle $2\alpha\pi$ transforms (1) into $e^{2(m+1)\alpha\pi i}$, which is a *proper* subset of it not containing $e^{2\alpha\pi i}$.

---

* This property has indeed been used by R. Dedekind (Was sind und was sollen die Zahlen, § 5) to define finite sets.

As this example is important for the general investigation given in this paper, it is useful to state the following property of the set (1).

The points (1) form a set which is dense everywhere on the unit-circle;

i.e. on every arc of the unit-circle there are points (1).

The factor $m\alpha$ in the exponent of (1) can obviously be replaced by

$$m\alpha - p \qquad \dots \quad (2)$$

where p is any integer.

The proposition therefore follows from the well-known fact that the numbers (2) form a set which for $m = 1, 2, \dots$ is dense everywhere in the interval (0, 1) —— and of course everywhere on the real axis. For the convenience of the reader, a proof of that statement will be given here.

Expand α as an infinite simple continued fraction, and let P_n, Q_n be the convergents; then for every odd n

$$0 < \alpha - \frac{P_n}{Q_n} < 1/Q_n^2$$

and therefore $$0 < \alpha Q_n - P_n < 1/Q_n.$$

As $\lim 1/Q_n = 0$, there exists for every $\varepsilon > 0$ such an n that $0 < \beta = \alpha Q_n - P_n < \varepsilon$.

Hence in every subinterval of (0, 1) of length $\geqslant \varepsilon$, there is at least one of the points $\beta, 2\beta, 3\beta, \dots$ and therefore a number (2). As this holds for every $\varepsilon > 0$, the set formed by the points (2) is dense everywhere in the interval (0, 1). Hence the set formed by the points (1) is dense everywhere on the unit-circle.

The corresponding statement holds for the set of the points

$$r\, e^{(2m\alpha+c)\pi i} \qquad \dots \quad (1')$$

as (1′) is obtained from (1) by a similarity transformation.

§ 2. *Transformations carrying M into a congruent proper subset.*

Let M be a bounded plane point-set which has congruent proper subsets. As it has been shown in § 1, sets of this kind exist.

By $\mathbf{A}, \mathbf{B}, \dots$ ... (3)
(with or without indices) will be denoted rigid motions of the plane carrying M into a proper subset; these form a system Σ.

$$\mathbf{A}(S) \qquad \dots \quad (4)$$

means the set in which a set S is carried by the motion **A**.

For every S, **A**(S) is congruent to S.

Especially

$$\mathbf{A}\,(\mathrm{M}) \subset \mathrm{M},\ \mathbf{B}\,(\mathrm{M}) \subset \mathrm{M}.$$
$$\mathbf{B}\,(\mathbf{A}(\mathrm{M})) \subset \mathbf{B}\,(\mathrm{M}) \subset \mathrm{M}.$$

On the other hand **B** (**A**(M)) is congruent to **A** (M) and therefore to M.

If one therefore uses at first the transformation **A**, and then **B**, one gets again a transformation of the system Σ, which will be called **BA**. Thus

$$\mathbf{BA}(\mathrm{M}) = \mathbf{B}\,(\mathbf{A}(\mathrm{M})) \subset \mathrm{M}.$$

As usual, **AA** is replaced by $\mathbf{A}^2$ etc.

If a transformation **A** out of Σ is of the type of a reflection i.e. if its determinant is equal to -1, then $\mathbf{A}^2$, which also belongs to Σ is of determinant $+1$. There is therefore no loss of generality if one restricts the investigation to rigid motions in the *stricter sense*, *i.e. to orthogonal transformations of the plane with determinant* equal to $+1$. These motions are either translatory motions or rotations.

With a suitable system of co-ordinates, any translation is expressed by

$$x' = x + c, \quad y' = y.$$

As M is bounded, there exists an upper limit for the values of x, say x_0; this value is transformed into $x_0 + c$. For $c = 0$, the transformation becomes the identity; for $c \neq 0$, it implies the existence of points not belonging to M. Hence a parallel-displacement cannot carry M into a proper subset of M.

The transformations (3) must therefore consist of rotations. Let $2\alpha\pi$ be the angle of rotation corresponding to a rigid motion **A**.

If α is rational, say $\alpha = p/q > 0$ and p being an integer then $\mathbf{A}^q$ is a rotation through $2p\pi$; that is the identity, contrary to the supposition that **A** and therefore its positive powers carry M into a proper subset of M. Hence α is an irrational number. Let O be the centre of the rotation **A**, and $re^{c\pi i}$ be an arbitrary point of M, then

$$\mathbf{A}^n(re^{c\pi i}) = r\,e^{(2n\alpha + c)i\pi}$$

Hence every point (1′) belongs to M, and every point of the circle about O with radius r is a limiting point of M. By adding to M all its limiting points, one gets the *closure* [M] of M. From the consideration cited above it follows that [M] consists of circles about O, and that therefore the smallest convex figure including M is a circle about O, if O is the centre of a rotation carrying M into a proper subset of it. Hence there cannot exist more than one such centre. *The rigid motions (in the strict sense carrying M into a proper subset of itself have all the same centre* O*; the closure [M] of* M *consists of entire circles about* O.)

Two rotations $\mathbf{A}$ and $\mathbf{B}$ about the same centre O through angles $2\pi\alpha$ and $2\pi\beta$ taken one after the other form a rotation through $2\pi(\alpha+\beta)$.

Hence

$$\mathbf{AB} = \mathbf{BA}.$$

The system Σ *is composed of commutative rotations.*

Consider any particular rotation $\mathbf{A}$ out of Σ, then there exists at least one point P which belongs to M but not to $\mathbf{A}$ (M). The inverse rotation $\mathbf{A}^{-1}$ carries P into a point not belonging to M; hence $\mathbf{A}^{-1}$ does not belong to Σ. The system Σ therefore does not form a group, but a *commutative half-group.*[*] The elements of this half-group are rotations about a particular point O.

§ 3. *The half-group* Σ *and the group G.*

Consider the rotations

$$\mathbf{A}\,\mathbf{B}^{-1} \qquad \dots (5)$$

where $\mathbf{A}$, $\mathbf{B}$. . . belong to the half-group Σ.

As
$$(\mathbf{A}\,\mathbf{B}^{-1})^{-1} = \mathbf{B}\,\mathbf{A}^{-1}$$

$$(\mathbf{A}_1\,\mathbf{B}_1^{-1})\,(\mathbf{A}_2\,\mathbf{B}_2^{-1}) = (\mathbf{A}_1\,\mathbf{A}_2)\,(\mathbf{B}_1\,\mathbf{B}_2)^{-1}$$

the rotations (5) form a (commutative group)[†] which will be denoted by G. As $\mathbf{A} = (\mathbf{A}^2)\,\mathbf{A}^{-1}$, every element of Σ is an element of G,

i.e. $$\Sigma \subset G.$$

Let P be a point of M. Consider the set M (P) of those points of M which are obtained from P by a suitable transformation out of G. It may be mentioned that not every element of G carries necessarily P into an element of M. Every element of Σ does, but there are elements of G not belonging to Σ – e.g. the identity and for particular P, even every element of G – which have the same property. Let Q be a point of M (P) and R be a point of M (Q) then

---

[*] A half-group is a system in which an associative composition of every pair of elements can be carried out.

[†] It is not necessary to check the associative law as the elements are transformations.

$$Q = \mathbf{A}\,\mathbf{B}^{-1}(P),\ R = \mathbf{A}_1\mathbf{B}_1^{-1}\,(Q);$$

hence $$P = \mathbf{B}\,\mathbf{A}^{-1}(Q),\ R = (\mathbf{A}\mathbf{A}_1)\,(\mathbf{B}\mathbf{B}_1)^{-1}(P).$$

The point-set M(P) is therefore uniquely determined by any one of its points

$$M(P) = M(Q). \qquad \dots \quad (6)$$

Furthermore $\mathbf{B}(Q) = \mathbf{A}(P)$ is a point of M. If on the other hand this equation holds for two points P and Q of M and two rotations **A** and **B** of Σ, then (6) holds. Hence two points of M belong to the same *class* M(P) if they can be transformed into the same point by suitable rotations of Σ.

If every element of G transforms P into a point of M, then the same holds for every point of M(P). In this case every element of G – and therefore every element of Σ – transforms M(P) into itself. Hence in this case no point of M(P) is *lost by the rotations of the half-group Σ. If, on the other hand*, one of the rotations (3) transforms M(P) into a set not containing P, then it transforms M(P) into a proper subset of M, and therefore into a proper subset of M(P). As M(P) = M(Q), every point Q of M(P) must be lost by a suitable rotation (3). The points M(P) are situated on the same circle about O. Certain considerations may be restricted to these constituent sets M(P), but it must be mentioned that the half-group of rotations transforming M(P) into a *proper* subset, may be a subset Σ′ of G which is different from Σ. Putting M(P) for M and Σ′ for Σ one gets again a subdivision of M(P) into classes and this procedure may not end after a finite number of steps.

§ 4. *A set* M *corresponding to a given* Σ.

The group G is a commutative group in which every element except the unit-element (the identity) is of infinite order. The power of this group is at most equal to the power of the continuum. Σ is a subset of G which forms such a half-group that every element of G is the quotient of two elements of Σ, and such that Σ does not contain the inverse of any of its elements. It will be proved that these conditions are sufficient too.

For this purpose, one has to apply the fundamental theorem of the general theory of point-sets that every set can be *well-ordered*. This theorem will be applied firstly to the set of all the real numbers, and then to any half-group Σ satisfying the above conditions.

Thus, let the real numbers be well-ordered in such a manner that the first element is the number 1.

$$\gamma_0 = 1,\ \gamma_1,\ \gamma_2,\ \dots .$$

Subdivide this system into two sets A and B, where B contains those and only those elements γ_μ which are linearly dependent on a finite number of *preceding* elements with rational coefficients.

Hence 1 belongs to A, whereas every rational number $\neq 1$ belongs to B. Both the sets A and B are well-ordered sets.

That every element of B is linearly dependent on elements of A is seen in the following manner. If there were elements in B not dependent on elements of A, then there must be a first one say β_0. This element is dependent on preceding elements

$$\beta_0 = c_1\gamma_{\nu_1} + \dots + c_l\,\gamma_{\nu_l}$$

Some of the elements γ_{ν_k} may be elements of A; those which belong to B are of a lower order than β_0 and therefore depending on elements of A. Hence β_0 depends on A; therefore every element of B depends on a finite number of elements of A. Every real number can therefore be represented in the form

$$c_0 + c_1\,\alpha_{\nu_1} + \dots + c_n\,\alpha_{\nu_n} \qquad \dots (7)$$

where the coefficients are rational numbers, and $\alpha_{\nu_1}, \dots, \alpha_{\nu_n}$ are irrational elements of A. The representation (7) is unique; for if a real number has two representations of this kind, the difference

$$0 = d_0 + d_1\alpha_{\mu_1} + , \dots + d_m\alpha_{\mu_m}$$

must vanish. As $1, \alpha_{\mu_1}, \dots, \alpha_{\mu_m}$ are by definition linearly independent, the coefficients are all equal to zero.

Thus the elements of A form a *basis* of the set of the real numbers. Every real number can be represented with this basis in one and only one manner in the form (7). The indices are not all integral numbers, but transfinite numbers; the basis forms a set of the power of the continuum.

Again let $\sum$ be a half-group being a subset of a commutative group G, and not containing the inverse element of any of its elements. The power of G is supposed to be at most equal to the power of the continuum, and its elements—except the unit-element

1—to be of infinite order. As 1 is its own inverse element, it cannot belong to Σ. Elements $\delta_1, \ldots, \delta_m$ are said to be *independent* if

$$\delta_1^{c_1} \cdots \delta_m^{c_m} = 1 \qquad \ldots (8)$$

implies $c_1 = \cdots = c_m = 0$. If however in (8) $c_m \neq 0$, then δ_m is *dependent* on $\delta_1, \ldots, \delta_{m-1}$. Let

$$\delta_1^{c_1} \cdots \delta_m^{c_m} = \delta^n = \varepsilon^n,$$

then $(\delta\varepsilon^{-1})^n = 1$. As an element of G which is of finite order must be equal to 1 it follows that δ and ε are equal. It is therefore not ambiguous to use fractional exponents[*] and to write

$$\delta = \delta_1^{d_1} \cdots \delta_m^{d_m}, \text{ where } d_k = c_k \, n \qquad \ldots (9)$$

In multiplying two products with fractional exponents one has to add simply corresponding exponents. It must however be mentioned that not for every system of fractions $d_1, \ldots, d_m$, the formula (9) furnishes an element δ of G.

In exactly the same manner as the existence of a well-ordered basis of the set of the real numbers has been shown above, one can prove the existence of a *basis of* Σ, i.e. a well-ordered set of elements

$$\mathbf{A}_1, \mathbf{A}_2, \ldots \qquad \ldots (10)$$

by the help of which every element of Σ can be represented as a finite product

$$\mathbf{A}_1^{d_1} \cdots \mathbf{A}_s^{d_s} \qquad \ldots (11)$$

As (10) forms a subset of Σ, and therefore of G, its power is at most equal to the power of the continuum. Hence to every element (11) there corresponds an irrational number

$$\gamma = d_1 \alpha_{\nu_1} + d_2 \alpha_{\nu_2} + \cdots + d_t \alpha_{\nu_t} \qquad \ldots (12)$$

and furthermore a rotation about O, through the angle

$$2\pi\gamma \qquad \ldots (13)$$

As the irrational numbers a_{ν_k} are independent, to different elements (11) there correspond different numbers (12), and—as these numbers are irrational—different rotations (13). To the product of two elements of $\sum$ there corresponds the sum of the corresponding numbers (12) and therefore the rotation composed by the two corresponding rotations. The half-group $\sum$ is therefore *isomorphic* with the system of the rotations through the angles $2\gamma\pi$. Consider now the points

---

[*] See the author's pamphlet: Abelsche Gouppen mit abzählbaren Elementen (Leipzig 1917).—For recent investigations on groups of this kind, See **R. Baer**: Abelian groups without elements of finite order, Duke Mathematical Journal, 3 (1937).

$$e^{2\pi\gamma' i} \qquad \ldots \quad (14)$$

where γ' takes all the values (12) *and* the value zero. These form a set, say M, situated on the unit-circle. As the inverse of any element of $\sum$ does not belong to $\sum$, the negative of any number (12) is not a number (12). For $\gamma' \neq 0$, the point $e^{-2\pi\gamma' i}$ is therefore not a point of M. By every rotation (13) the set M is transformed into a congruent set, which is a subset of M as the sum of two numbers γ' is again a number γ'. This subset does not contain the point 1, as it would be the image of a point $e^{-2\pi\gamma i}$ which is outside M. Hence to every element of $\sum$ there corresponds a rotation carrying M into a proper subset of itself. On the other hand a rotation of this kind transforms 1 into a point (14), where $\gamma' \neq 0$ as the rotation cannot be the identity; hence it is a rotation (13). The given half-group $\sum$ is therefore isomorphic with the half-group of the rotations carrying M into a proper subset of itself. If P is an arbitrary point of M, then obviously

$$\mathrm{M}\ (\mathrm{P}) = \mathrm{M}. \qquad \ldots \quad (15)$$

From this formula it is evident that the type of M is in general not uniquely defined by M. It may be mentioned without proof that even if the half-group of rotations (13) is given there exist on the unit-circle, in general, essentially different sets M satisfying the condition (15).

§ 5. ***Measure of the lost points.***

Again, let M be transformed into a proper subset M′ by a rigid motion, and let L be the set of the "lost" points. If M is measurable then M′ and L are measurable and from

$$|\mathrm{M}| = |\mathrm{M}'| + |\mathrm{L}|$$

and $$|\mathrm{M}| = |\mathrm{M}'|$$

it follows that $|\mathrm{L}| = 0$.

These equations hold equally whether the measure is taken in Jordan's or in Lebesgue's sense, and whether the measure is an areal measure or—in case M be situated on a single circle—an one-dimensional arc-measure.

If M is non-measurable the consideration just stated furnishes only a statement on inner measure:

$$\underline{M} > \underline{M}' + \underline{L}$$

$$\underline{M} = \underline{M}'$$

Hence $$\underline{L} = 0.$$

Is it possible that the outer measure $\overline{L}$ is positive? Here one has to distinguish between the different definitions of measurement. In Jordan's sense the outer measure of a point-set is already positive when the set is dense everywhere in an interval. Sets M which lose subsets dense everywhere on the unit-circle can easily be constructed, e.g. the point-set $e^{2\pi\gamma i}$, where

$$\gamma = m\sqrt{2} + n\sqrt{3}, \qquad (m,n = 0,1,2,3 \cdots)$$

Indeed, by the rotation through the angle $2\pi\sqrt{2}$, this point-set is transformed into a subset thereby losing the points $e^{2\pi n\sqrt{3}i}$ for $n = 0,1,2 \cdots$, and these points are dense everywhere on the unit-circle. If we alter this example slightly by projecting the points on the circle from the centre, the lost points will be dense everywhere inside and on the unit-circle, and the outer areal Jordan-measure will be positive. The outer Lebesgue's measure however is zero. Thus the paradoxical result seems to be a consequence of the imperfection of Jordan's method of measurement.

It will however be proved now that there exist point-sets M which lose by certain rotations sets of positive outer Lebesgue's measure. Let $$1, \alpha_1, \alpha_2, \cdots$$
be a basis of the real numbers as in § 4. Then the angles of rotation are *uniquely represented by* $2\pi\gamma$, where

$$\gamma = c_0 + c_{\nu_1}\alpha_{\nu_1} + \cdots + c_{\nu_m}\alpha_{\nu_m}$$

the coefficients being rational, and $0 \leqslant c_0 < 1$.

The points $e^{2\pi\gamma i}$ will now be distributed into an enumerable number of congruent subsets without common points

$$M_i, \quad i = 0, \pm 1 \pm 2, \ldots$$

which are defined by the inequality

$$i \leqslant c_1 < i+1.$$

obviously M_i is transformed into M_{i+n} by a rotation through $2\pi n\alpha_1$.

This shows the congruence of these sets. If they are measurable in Lebesgue's sense, they would be of equal measure, say $|M_0|$ and as a consequence of *a fundamental theorem on* Lebesgue's measure,

$$2\pi = |M_0| \sum_{-\infty}^{+\infty} n$$

which is impossible. Hence the sets are not measurable, and they have therefore a positive outer-measure.

Let now M be the set of those points $e^{2\pi\gamma i}$ for which $c_1 > 0$. By the rotation through the angle $2\pi\alpha_1$, the set M is transformed into a proper subset thereby losing the point set M_0 which has a positive outer Lebesgue's measure.

Ghoom (Darjeeling)
13-11-39.

GLEANINGS

Such a revolution in the rudiments of mathematical thinking does not go unchallenged. Brouwer's radical move to the left is speeded by an outraged roar from the reactionary right. "What Weyl and Brouwer are doing [Brouwer is the leader, Weyl his companion in revolt] is mainly following in the steps of Kronecker," according to Hilbert, the champion of the *status quo*. "They are trying to establish mathematics by jettisoning everything which does not suit them and setting up an embargo. The effect is to dismember our science and to run the risk of losing a large part of our most valuable possessions. Weyl and Brouwer condemn the general notions of irrational numbers, of functions—even of such functions as occur in the theory of numbers—Cantor's transfinite numbers, etc., the theorem that infinite set of positive integers has a least, and even the 'law of excluded middle,' as for example the assertion: Either there is only a finite number of primes or infinitely many. These are examples of (to them) forbidden theorems and modes of reasoning. I believe that impotent as Kronecker was to abolish irrational numbers (Weyl and Brouwer do permit us to retain a torso), no less impotent will their efforts prove today. No! Brouwer's program is not a revolution, but merely the repetition of a futile *coup de main* with old methods, but which was then undertaken with greater verve, yet failed utterly. Today the state [mathematics] is thoroughly armed and strengthened through labours of Frege, Dedekind, and Cantor. The efforts of Brouwer and Weyl are foredoomed to futility."

To which the other side replies by a shrug of the shoulders and goes ahead with its great and fundamentally new task of re-establishing mathematics (particularly the foundation of analysis) on a firmer basis than any laid down by the men of the past 2500 years from Pythagoras to Weierstrass.

E. T. BELL, *Men of Mathematics.*

GENERALISED GEOMETRY AND PHYSICAL THEORIES †

BY

B. S. MADHAVA RAO, *Department of Mathematics, University of Mysore, Bangalore*

SUMMARY

(1) INTRODUCTION: —Relations between mathematics and physics—relation between geometry and physics—Relativity and geometrisation of physics—Quantum mechanics and abstract space—non-linear field theories.

(2) CLASSICAL MECHANICS:—Hamilton's characteristic function—configuration space—geometry of dynamics—principles of least action and least curvature—Phase space—dynamics and topology.

(3) RELATIVITY AND COSMOLOGY:—Unitary field theories—Stellmacher's work—The approximation method—Theory of supernovae.

(4) QUANTUM MECHANICS:—Hilbert space—Logic of quantum mechanics—Electron wave equation in general relativity and generalised spaces.

(5) ELECTROMAGNETIC FIELD THEORIES:—Quantum Electrodynamics—Born's field theory—Rosen's theory—Dirac's classical theory of the electron—Work of March and Foradori.

(6) NUCLEAR PHYSICS:—Mesotron theory—Born's reciprocity—Theory of Proca and Goudsmit—Wave geometry of the Japanese School.

1. Introduction

In the course of the development of modern theoretical Physics, the notions of generalised geometry have played a very prominent part. It is true that, of all branches of mathematics, Analysis has for a very long time been the principal weapon in the armoury of the theoretical physicist, and it is only recently that other branches are coming to be employed more frequently in the building up of the foundations of Physics and the erection of its superstructure. As examples in point one might mention the Birkhoff-Neumann logic of quantum mechanics (1), the algebra associated with it (2), the applications of group theory in wave mechanics and molecular physics (3, 4), considerations of probability and general statistics (5), and the weird programme of "*arithmetisation of Physics*" (6) initiated by Milne in his cosmological speculations.

---

† Paper presented to the symposium on "Generalised Geometry including Relativity and Field Theories", Eleventh Conference of the Indian Mathematical Society, Hyderabad, Deccan, 22nd December 1939.

The connections between geometry and physics have, however, always been intimate. The idea that physical space is identical with three-dimensional euclidean space has dominated classical physics for a long time. With the advent of non-Euclidean geometry, an era of revolution was ushered in geometry which came to be placed on an axiomatic basis, and there appeared a temporary divorce between physics and geometry. The theory of relativity was responsible for again bringing about an intimate connection between them by stimulating investigations regarding the relationship of natural geometry, and abstract geometry. In fact, the golden age of generalised geometry—the subject to be dealt with in this symposium—might be truly said to have begun with the formulation of the general theory of relativity by Einstein, although it must be realised that this theory would probably not have seen the light of day had it not been for the creation of the absolute differential calculus by Ricci and Levi-Civita. For the first time, we have become familiar, through general relativity, with the fact that geometry, in so far as it introduces the determination of measure in the space-time world, has an essentially empirical character. This knowledge, culminating in the researches of Einstein on gravitational phenomena, and Weyl's attempt at the unification of gravitation and electro-magnetism, has given birth to the movement known as the "*geometrisation of Physics*" which aims at the construction of unitary theories reducing all physical phenomena of the field to a geometry of space-time or a space-time-something else, and has also been responsible for the opening up of a new field of geometry—the geometry of linear connection.

With the rise of Quantum Mechanics, the geometry of abstract spaces has reaped a rich harvest in Physics. The difficulties connected with the applications of quantum mechanics to field theories, with the positron theory, with the explanation of the mysterious specifically nuclear forces between nuclear particles, and with the mesotron theory have given rise to a number of geometrical theories of a Riemannian or non-Riemannian character, and even of the general topological type. Even the staid old classical dynamics is being dominated by these newer concepts of generalised geometry.

It might be that the physical ideas behind many of these theories are not of much significance in unravelling the mysteries of physics. But, they are nevertheless interesting in indicating methods of mathematical formulation which may probably be adopted in future successful theories. This is one of the reasons which have prompted the presentation of some of these theories in this address.

2. Classical Mechanics

The first significant geometrical idea introduced into classical mechanics was probably the most fruitful conception of Hamilton that the dynamical equations could be deduced on an optical analogy as the motion of a wave front from one position to a neighbouring position (7). This is really the basis of the transformation theory of classical dynamics according to which the canonical equations of motion simply express the fact that the whole course of a dynamical system can be regarded as the gradual self-unfolding of an infinitesimal contact transformation.

The most obvious geometrical method of expressing the motion of a dynamical system is to associate a "Configuration Space" M_n every point $(q_1, q_2 \ldots . q_n)$ of which denotes a state of the system, the motion being represented as a trajectory in this space. An alternative method is to think of a space of the (q_n, t) whose "world lines" denote the motion of the dynamical system. Using these ideas all the theorems of generalised dynamics can be expressed in a geometrical form.

Thus either in M_n or M_{n+1} the totality of world-lines or trajectories constitutes a group of $2n$-parametric curves. Through any point of M_{n+1} there passes a n-parametric bundle of curves only one of which passes through a neighbouring point, the bundle thus constituting a field. In M_n, however, there are ∞^1 trajectories through a point in a specified direction.

If the system be holonomic and without moving constraints with the kinetic energy in the form

$$T = \tfrac{1}{2}\, a_{ij}\, \dot{x}^i\, \dot{x}^j$$

we may assign to M_n the kinematical line-element (8)

$$ds^2 = a_{ij}\, dx^i\, dx^j$$

and thus obtain a Riemannian space R_n defined topologically and metrically by the dynamical system. The interest of this representation is twofold in that it not only gives a geometrical interpretation to a dynamical problem but also gives a dynamical interpretation to multi-dimensional spaces or to spaces with a topology and metric not realisable in a Euclidean space. To consider two simple special examples, it might be mentioned that the dynamical system, consisting of a disc which can turn about an axis perpendicular to its plane with its centre at the same time free to move along a line in the plane, gives a "configuration space" which is a flat space of two dimensions having the connectivity of a cylinder. Again, the system of two discs rotating independently gives rise to a flat space of two dimensions having the connectivity of an anchor ring (9).

Coming back to the general geometry of R_n, it is to be noticed that all vectors have covariant and contravariant components; in particular the covariant components of the velocity vector are the generalised momenta. Lagrange's equations of motion can be written in the form

$$\ddot{q}^r + \left\{ {r \atop st} \right\} \dot{q}^t \dot{q}^s - Q^r$$

considering the trajectories through a given point with specified directions $(dq_r/ds)_0$ those are determined by the initial velocity of projection $v_0 = (\dot{s})_0$, the variation of which alters merely the *geodesic curvature* of the trajectories This curvature K_g can be obtained in the form

$$K^2_g = \frac{1}{\dot{s}^4} \left\{ g_{sr}\, Q^s Q^r - \left(Q_r \frac{dq_r}{ds} \right)^2 \right\}$$

and the equations of motion reduce to

$$\frac{d^2 q^r}{ds^2} + \left\{ {r \atop pt} \right\} \frac{dq^p}{ds} \frac{dq^t}{ds} = \frac{1}{\dot{s}^2} \left\{ Q^r - Q_p \frac{dq^p}{ds} \frac{dq^r}{ds} \right\}$$

which shows that the geodesic lines (with $K_g = 0$) are the limiting positions of the trajectories for $s = \infty$. (10). In the case where the external forces are absent, K_g is identically zero and the trajectories are the geodesic lines of the Riemannian space. It might also be mentioned that the K_g introduced here is the same as the curvature of the Gauss-Hertz law.

For systems with constant energy we might consider, instead of the kinematic line-element, the *action line-element*

$$ds^2 = 2\ (h - v)\ T dt^2 = g_{ij}\ dq^i\ dq^j$$

and the principle of least action is equivalent to stating that the curves of natural motion are the geodesics in this Riemannian space. Hamilton's variation principle can be shown to be the same as this principle translated to the space-time (M_{n+1}).

Besides the notion of configuration space, it is also possible to associate with a dynamical system a "*phase-space*" $\sum$ of $2n$ - dimensions every point of which corresponds to the choice of n co-ordinates of position and the n conjugate momenta. Equally useful is the setting up of the $(2n+1)$ dimensional phase-space $\sum'$ including t as well. In either case the equations of motion can be considered as laws of propagation inducing a steady fluid motion in the phase-space. The existence of the integral invariant of order $2n$.

$$\int\int \cdots \int \delta p_1 \cdot\cdot\, \delta p_n\, \delta q_1 \cdot\cdot\, \delta q_n = \text{const}$$

for the canonical system of equations is equivalent to the statement of conservation of volume in this fluid motion. With this representation it is possible to give a geometrical meaning to the Hamilton-Jacobi theory, as well as to the dynamical theorems based on Lie's transformation group theory.

The most fruitful results arising as consequences of the notion of the phase-space are perhaps the investigations relating to dynamical motions "im Grossen". Such questions raised for the first time by Poincaré relate to the totality of all possible forms of motions and the complete course of any such particular form. Since the manifold of configurations of a dynamical system correspond to a set of points in phase-space, the above type of questions requires topological methods for its solution. These have been developed by Birkhoff (11) who has considered the simplest types of motion chosen in as simple a way as possible. With the growth of topology in recent years there has also been consequent development in this branch of the subject specially with regard to a type of motions known as *transitive*. A dynamical system is *regionally transitive* if there exists a motion such that the points of phase-space defined by this motion constitute a set which is everywhere dense in the phase-space, and such a motion is called *transitive*. If the set of points be measurable and either of measure zero or measure equal to that of the phase-space, the system is said to be *metrically transitive*. As a typical theorem in this field might be mentioned a result due to Hedlund (12) that the systems of geodesics on a certain set of surfaces which are closed, orientable, of arbitrary genus greater than one, and of constant negative curvature are metrically transitive.

3. Relativity and Cosmology

The general theory of relativity is one of the landmarks in the history of the relations between geometry and physics. It is not proposed to deal here with the classical aspects of either the special or general theories of relativity, but it would be worthwhile to say a few words regarding recent work in the domain of unitary field theories seeking to unify gravitational and electromagnetic phenomena into a single geometrical scheme. Of the several attempts so far made in this direction two of the simplest and most natural ones are those due to Weyl (13) and Kaluza and Klein (14) using respectively the notions of gauge-invariance, and a fifth dimension. Even the celebrated attempt of Einstein-Mayer (15) introducing the notion of "teleparallelism" must be considered rather artificial. Recently, however, Einstein and Bergmann (16) have proposed a slight modification of the Kaluza theory which might be noticed here.

The Kaluza-Klein theory introduces a fifth dimension in attempting to derive a unitary theory connecting gravitation and electricity. Einstein has attempted to generalise this theory by putting in physical concepts into the purely mathematical structure of Kaluza's theory.

The aim of this theory of Kaluza was to obtain some new physical aspect for gravitation and electricity by introducing a unitary field structure with the aid of a fifth dimension, the essential result being that such a five-dimensional structure could be built up so as to be equivalent to a four-dimensional structure *plus* a vector field which is the potential vector for the electro-magnetic field. This result, though elegant mathematically, was not productive of new physical ideas, and consequently many attempts were made to retain the essential formal results obtained by Kaluza without sacrificing the four-dimensional character of the physical space. But all such attempts have proved unproductive, and it appears impossible to formulate Kaluza's idea in a simple way without introducing the fifth dimension.

On the basis of these considerations, Einstein and Bergmann have now attempted to introduce the fifth dimension in a very effective manner without its being merely a sort of "catalytic agent" as in the Kaluza theory. To bring out clearly the generalisation proposed by Einstein, let us consider how Kaluza's five-dimensional structure is made equivalent to a four-dimensional one and a vector field. It can be shown that by a suitable characterisation of the 5-space with the metric,

$$d\sigma^2 = \gamma_{\mu\nu} dx^\mu dx^\nu \qquad (\mu, \nu = 0, 1, 2, 3, 4)$$

the components of the fundamental metric tensor can, by the choice of a special co-ordinate system, be reduced to ten functions g_{mn} and the four functions A_m $(m, n=1, 2, 3, 4)$ which do not depend on x^0. This reduction gives a four-dimensional description of the space, and the independence of the functions on x^0 shows the purely formal nature of the fifth dimension x^0 which just put in only to be taken out later. On Einstein's new theory it is shown that, with a suitable modification of the postulates of the 5-space, it is possible to make an exactly analogous reduction to g_{mn} and A_m with this difference that the components of g_{mn} are in general periodic functions of x^0. The A_m, however, is independent of x^0 as in the old theory. Remembering that g_{mn} is a four-dimensional metric tensor, this amounts to an intimate physical connection of the space-time with the new dimension. The x^0 which is put in at first is not taken out after the reduction but left behind so as to modify the 4-metric. The periodicity of the components of this 4-tensor in the new co-ordinate enables one to interpret physically the fifth dimension. In a very rough way, one could describe this as a sort of a phase, giving rise to the 5-dimensional space-time-phase. Since, however, this new co-ordinate is "dimensionless" there arises no contradiction with the empirical four-dimensional character of physical space.

From its very nature, the new theory is essentially complex in its physical aspects, and Einstein and Bergmann have given the derivation of the fourteen field equations starting from a variational principle, and also the indentities satisfied by the field equations. The theory involves four universal constants of which one corresponds to the gravitational constant involving a connection between the units of length and mass, another depending on the unit of length, while the remaining two are "genuine" universal constants which cannot be eliminated from the theory.

When looked at from the purely geometrical point of view the new theory introduces some very interesting features. The five-dimensional space defined by the metric is here closed with respect to one dimension, and this closed space will be represented by a space which is open and periodic with respect to this dimension. A point P of the physical space will be represented by an infinite number of points P, P′, P″ of the 5-space. This type of non-homeomorphic correspondence between general metric spaces is itself a rich mathematical concept capable of a large number of developments.

Apart from the above geometrical unitary theories, one might consider a more fundamental unitary tendency in attempts at unifying the notions of field and matter. All the field laws of physics, gravitational, electromagnetic and quantum, are structure laws which break down for very great concentrations of energy or where sources of the field are present. It has been obvious from the evolution of the history of physics that it cannot be built on the matter concept alone. The alternative to this is to find out if we could not reject the matter concept entirely, and build a pure field physics which explains nature by structure laws valid always and everywhere. This programme, has, so far, not been fulfilled in spite of the great advances made in quantum mechanics in recent times. With a few such unitary theories on the electromagnetic side we shall have occasion to deal later (§ 5). For the present, so far as it concerns gravitation, an important question that arises is whether the motion of matter considered as point-singularities of the field could be determined by the gravitational equations alone. This question has been answered in the affirmative by Einstein, Infeld and Hoffmann (17) who have set up an "approximation method" in the treatment of the problem, and have also applied it successfully to the relativistic problem of two bodies.

A geometrical treatment of the motion of singularities has also been attempted by Stellmacher (18) who has systematically applied Hadamard's method of characteristics and bicharacteristics to the field equations of gravitation as well as the Maxwell-Lorentz equations in the field of general relativity. Also the law according to which the singularities are propagated has been derived and this leads to the physically significant result that in the theory of characteristics it is essential to speak both of gravitational and electrical quanta. A treatment of gravitation alone appears meaningless from this standpoint. There exists however a remarkable relation between gravitational and electrical energy for which specially great stability results.

Coming to cosmological theories, it would perhaps be not quite wrong to describe them as relativistic theories of idealised geometrical models. Static and non-static models with specialised line elements and pressure-density distributions have been constructed and studied in great detail, and applied to the explanation of the fundamental phenomenon of nebular recession (19). The remarkable theory of Milne (20), which too had a cosmological origin, is in reality an alternative to the programme of geometrisation of physics by the substitution of kinematics for geometry.

As an example of a cosmological theory, which dispenses with the current idealised models and breaks new ground, might be mentioned a theory proposed by Zwicky (21) to explain the tremendous generation of energy in the recently discovered supernovae. Suggesting that the nova outburst is the result of the transformation of an ordinary star into a "*Collapsed neutron star*" with density of the order of 10^{14} gm/cm^3, i.e. of the order of density inside atomic nuclei, he develops the purely geometrical idea that the *very properties of space in highly collapsed stars may be radically different* *from the properties of ordinary space.* The classical Schwarzschild theory on which the usual stellar models are based even if combined with the fundamental result, $E = mc^2$ of special relativity is unsatisfactory in this case, in that it does not allow us to formulate clearly how the mass equivalent of the gravitational potential energy is to be distributed over space. This difficulty has its real root in the fact that gravitation is a *co-operative phenomenon,* and the general theory of relativity alone must be used to estimate such effects as the effective masses of large dense stars.

This theory of supernovae gives rise to another cosmological aspect of the theory of neutron stars viz., the conclusion of Stern (22) on the basis of statistical mechanics and thermodynamics that in a state of equilibrium between matter and radiation in the Universe, practically all matter would be found to have been transformed into radiation. The ratio of concentration of matter transformed into radiation to the concentration of matter in the ordinary forms is determined on Stern's Theory by the undefeatable magnitude $e^{6\times 10^7}$ or, in other words, with radius of Universe R equal to 6×10^{39} cm., and number of electrons $1{\cdot}2 \times 10^{78}$, not even a single electron could exist in R in statistical competition with radiation. As an escape from this, Zwicky has suggested consideration of matter, not in the ordinary forms, but in the clustering form as in neutron stars where the velocity becomes zero, and the gravitational packing energy of a particle of mass m may become equal to its original proper energy.

If the neutron-star hypothesis of the origin of super-novae can be proved, it will be possible to subject the general theory of relativity to a really crucial test. This theory, although profound and extremely satisfactory in its epistemological aspects, has so far practically not lent itself to any very obvious and generally impressive applications. It may thus be possible to remove the unfortunate discrepancy between the formal beauty of the general theory of relativity, and the meagreness of its practical applications.

4. Quantum mechanics

In geometry there have been two essentially distinct tendencies the abstract or axiomatic, and the constructive or analytical. Of these it might be said that the former has played the predominant role in enriching the conceptions of geometry. Thus the transition from Euclidean to Non-Euclidean geometry has been due to an abstract idea. Similarly the geometries of Riemann and Cartan, on the former of which Relativity has been based, are the results of generalising the abstract notions of curvature and torsion. In the development of the geometry of the infinite dimensional complex space which is the geometry appropriate to quantum mechanics, the suggestions of generalisation have, however, come from the purely analytical side, in fact have been based on algebraic analysis viz., the question of linear orthogonal transformations of a quadratic form due in its essence to Hermite. But the geometry itself can be developed in a purely axiomatic way (23). We can first construct axiomatically the ∞-dimensional real non-complex space starting with a simple generalisation of Euclidean space of n dimensions. The axioms of affine geometry, the idea of linear operators could first be introduced, and then the metric notion by the definition of the scalar product of two vectors, and the notion of orthogonality. The transition to the case $n \rightarrow \infty$ offers no difficulty from the abstract point of view, the associated difficulties arising only in an analytical treatment. The generalisation to the complex space follows on exactly parallel lines, and all the above notions can be retained intact. The last step is the reduction of the abstract geometry so obtained to the analytical form by the use of co-ordinates based on the notion of Hermitian symmetry thus showing that the complex metric geometry obtained is essentially Hermitian with its operators also Hermitian.

To show that this is the appropriate geometry for quantum mechanics it is necessary to mention that quantum mechanics shares along with classical mechanics, and classical electrodynamics the notion of phase space already mentioned (§ 2). According to this concept any physical system S is at each instant hypothetically associated with a "point p" in a fixed phase-space $\sum$ and this point is supposed to represent mathematically the "state" of S ascertainable by "maximal" observations [23, § 4 (1930)]. Furthermore the point p_0 associated with S at a time t_0 together with a prescribed "mathematical law of propagation" fix the point p_t associated with S at any later time t.

Thus in classical mechanics $\sum$ is a region of ordinary $2n$-dimensional space. In electrodynamics the points of $\sum$ can only be specified after certain functions such as the electro-magnetic and electro-static potentials are known, and hence $\sum$ is a *function space* of infinitely many dimensions. Similarly in quantum mechanics the points of $\sum$ correspond to "wave functions" which are complex and functions of space and time and hence $\sum$ is again a *function-space* and a complex space of infinitely many dimensions. The law of propagation is given by Schrodinger's equations just as Maxwell's equations give the law for electrodynamics. Using the analogy of fluid motion induced by the law of propagation, we might say that while the flow conserves volume in classical mechanics (§ 2), it conserves distances in quantum mechanics since the equations are unitary.

A difference might immediately be noticed between the physical quantities that represent a point in phase space in quantum mechanics and electrodynamics. While in the latter we might say that the laws of propagation i.e. Maxwell's equations determine the electromagnetic field, the interpretation in the former case is of a more abstract nature. The equations of quantum physics, while they are structure laws just like Maxwell's and Einstein's equations do not determine concrete things like the electromagnetic or gravitational fields but the abstract *probability waves*. Thus even in the case of the one particle problem, while we can speak in classical mechanics of the positions and velocities of the particle we can only consider, in quantum mechanics, probability waves in a three-dimensional continuum. For n particles the probability waves will be functions in a continuum of $3n$ dimensions, but the phase space is again a function space of infinity of dimensions.

Further analysis is however necessary to specify more exactly the nature of this function space so as to be in consonance with the principles of quantum theory. A penetrating analysis of the question has been given by J. v. Neumann (24) who has shown that, as a consequence of the equivalence of the matrix and wave mechanics points of view, this function space is isomorphic with a *Hilbert space* R_∞. The essence of the matrix point of view is the reduction of the Hamiltonian H to the diagonal form $S^{-1}HS$ which leads to the "eigenwerte" problem

$$\sum_{\nu} h_{\mu\nu}\, x_\nu = \lambda x_\mu$$

while the corresponding "eigenwerte" problem of wave mechanics is

$$H\psi(q_1 \cdot \cdot\, q_n) = \lambda\cdot\psi\ (q_1 \cdot \cdot\, q_n)$$

H being considered a differential operator. Thus we have two typical spaces, the discrete space Z with index values $(Z=1, 2, \ldots)$ and the continuous configuration space Ω of the physical system. The equivalence of the two theories does not mean the equivalence of Z and Ω, but that of the sequences $(x_1, x_2, \ldots\ldots)$ which are functions in Z and the wave function $\psi\ (q_1, q_2 \ldots\ldots q_n)$ which are functions in Ω i.e. of the function spaces F_z and F_Ω. Since the normalising condition $\sum_{\nu} |x_{\nu}|^2 = 1$ is assumed in F_2 this function space is the usual *Hilbert space*. That F_2 and F_{Ω} are isomorphs is merely an alternative way of expressing the well-known *Riesz-Fischer* theorem.

It has been noticed above that the ∞-dimensional complex space is the appropriate one for quantum mechanics, but the dimensionality axiom has not been rigorously examined. We might consider an abstract space R satisfying the following axioms:

A. R is a linear space.

B. In R a Hermitian scalar product is defined.

C.$^{(n)}$ There are exactly n linearly independent vectors in R. Then R would be the n-dimensional complex Euclidean space. If however the axiom $C^{(n)}$ be replaced by C^{∞} i.e. that there exists no such maximum number as n, then this really does not amount to an axiom at all, and hence in order that our space R should determine the Hilbert space R_{∞} it is necessary to impose two other axioms D and E viz.,

D. R is a complete space.

E. R is a separable space.

For R satisfying A, B and $C^{(n)}$, the axioms D, E follow as consequences, but for R to define R_{∞}, D and E are essential.

Having axiomatically defined the Hilbert space R_{∞} (F_2) its geometry could be developed analytically, and the notion of operators introduced. Lastly the introduction of the statistical postulate of probability amplitudes completes the geometrical scheme of quantum mechanics, and the correspondence between the geometrical magnitudes and their physical interpretation.

It will be evident from the above considerations that the foundations of quantum mechanics are essentially of a geometrical character, and this conclusion has been further strengthened by the work of Birkhoff and J. v. Neumann (2) who have examined the logic of quantum mechanics, and built up a calculus of propositions for quantum mechanics on the basis of the facts that all operators are Hermitian, and all Hermitian-symmetric operators in Hilbert space correspond to observables. A careful examination of this calculus has led to the conclusion that it has the same structure as an *abstract projective geometry*, while in classical mechanics the experimental propositions constitute a *Boolean algebra*.

Amongst one of the unsolved problems of quantum mechanics is its fusion with the theory of relativity. In all arguments about the transition from classical to quantum physics, only the pre-relativistic description is used in which space and time are treated differently. If, however, we try to begin from the classical description as proposed by the relativity theory then our ascent to the quantum problem is much more complicated. A partial solution of this problem is contained in the relativistic wave equation of the electron constructed by Dirac. This has enriched the concepts of geometry by showing that Lorentz invariance and tensor invariance are not necessarily identical, the former being more suitably expressed in terms of Spinors which play such an important part in the geometrical theory of projective relativity. The Dirac wave equation has also been the starting point of a series of researches of the highest importance to generalised geometry. A general invariant representation of Dirac's wave equation of the one-electron problem on the basis of Einstein's general relativity was first given by V. Fock (25) who investigated the wave-mechanical behaviour of the motion of an electron in an arbitrarily specified electro-magnetic and gravitational field. This theory has been further developed by Schrodinger (26), Schouten (27), Bargmann (28), Infeld and Van der Waerden (29). The representation of a corresponding wave equation for Einstein-Mayer's unitary field theory has been investigated by Pauli and Solomon (30), and, in a different mathematical way by Schouten and Van Dantzig (31) and also by Pauli (32). In the same category might be placed the work of Dirac (33) who has attempted to set up electron wave equations in de-Sitter space and conformal space. The geometrical interest in the extension to various spaces lies in the fact that the equations of atomic physics have to be so formulated that they form a scheme which is invariant for groups of transformations which carry over the space into itself. Nearly all the more general spaces have only trivial groups of operations of this nature, but the above two spaces are exceptions. Defining de-Sitter space as the surface of a four-dimensional "sphere" (of a hyperbolic character in one dimension) imbedded in five-dimensional space, Dirac has shown that all the principal equations can be expressed in tensor form involving indices taking on five values. For conformal space defined as the four dimensional surface of a hyper-quadric in five dimensional projective space, it is found, on the other hand, that the tensor-equations require the use of six valued indices, and, moreover, not all equations of physics but only those that are invariant under conformal transformations of the ordinary space-time can be generalised to conformal space. In particular it is found that the electron wave equation cannot be so expressed while it is possible for Maxwell's equations giving an indirect proof of the result due to Cunningham and Bateman (34) that Maxwell's equations are invariant under conformation transformations.

From the physical point of view, however, the above generalised theories introduce additional terms in Dirac's equation the deviations of which from the ordinary equation are so small as to be hardly within the range of experimental determination.

5. Electro-magnetic field theories

We have mentioned above one of the fundamental difficulties besetting quantum theory viz., the failure to bring about a general and natural fusion between relativity and quantum theory. Another equally important difficulty is the question of the interaction between elementary quanta of matter and the electro-magnetic field, or the so-called quantum electrodynamics. In the many particle problem, we have seen that the description of quantum phenomena could be achieved by considering probability waves in space of $3n$ dimensions. But for the quantum description of the field it would be necessary to consider a probability wave with an infinite number of dimensions, and this transition from the classical field concept to the quantum concept is a very difficult step, and all the theories so far developed in this direction are to be regarded as unsatisfactory.

Let us consider in a little more detail one such theory which is provisionally widely accepted viz., the Heisenberg-Pauli Theory (35) which includes as a particular case all other previous theories including those of Dirac and Fermi (36). This theory has been shown to be eminently successful within well-defined limits (37) in explaining phenomena of the quantum theory of radiation. In essence, this theory is a generalisation of Dirac's theory of radiation in what it considers a quantum theory of the total field including the electrostatic. In the division of the field into plane waves (i.e. spatial Fourier analysis) into transversal and longitudinal waves, it can be shown that the first determine the characteristic light waves and the second the static interactions the latter replacing the longitudinal waves which do not lead to any new results on quantisation.

The difficulties of this quantum-electrodynamics appear to be of two sorts. The first can be indicated by considering, for the moment, the force-free particles of the Dirac's wave equation, and by coupling of these electrons with the electromagnetic field we immediately have difficulties connected with negative energy states. Even for a quantum electrodynamics *in vacuo* the results are not valid for wave lengths $\lambda < \lambda_0$ ($= h/mc$), the Compton wave-length; and if valid for $\lambda < h/mc$, the field of interaction must be very weak. The other difficulty is that the theory leads to an infinite self-energy of the electron, and wave lengths $\lambda < r_0$ ($= e^2/mc^2$) are excluded from the theory. The second difficulty is perhaps not exactly quantum-theoretic in its origin; in fact, the classical (Lorentz) theory of the electron yields, when the electron radius goes to zero, an infinite quantity, the 'electro-magnetic mass". In other words, a point electron cannot, on account of the infinite inertia of its associated electro-magnetic field, be accelerated by a finite external force. In the Heisenberg-Pauli theory also which starts with point electrons one obtains for the electrons an infinite inertial mass (38).

To get over this second difficulty several theories have been proposed. One such theory is Born's non-linear field theory (39) which, although it has not been able to achieve the purpose for which it was intended, has nevertheless added new ideas to quantum electrodynamics and given a beautiful solution to the old classical problem of the structure and inertial mass of the electron. This theory is unitary in its nature in that it considers an electron as a point singularity in an electrodynamic field, and Maxwell's equations of the field are modified so as to make the energy of the field around the singularity representing the electron's self-energy a finite quantity. It has been known for quite a long time that in the self-energy problem the infinity could be got over by the assumption of an extended electron, but then the theory would become non-invariant relativistically. For, suppose we introduce a form function $D(x, y, z)$ to give the spatial extension of an electron such that the function is different from zero only in the inside of a small region; then by changing into another co-ordinate system the time t appears in this function, and it becomes impossible to formulate the field laws invariantly. In addition to this difficulty, the assumption of an extended electron would introduce a good deal of arbitrariness (in the choice of charge distribution) in the theory and would also go against the postulate of atomistic constitution of electricity. Born's field theory overcomes all these objections since the equations of the theory are relativistically invariant. It might also be noticed that Born's field theory is also gauge-invariant which condition has been shown to be essential by Weyl for any satisfactory field theory. This is achieved in Born's field theory since the Lagrangian used in the variational principle on which the whole theory is based contains only the field strengths and not the potentials.

Considering first the classical part of the theory, one starts with the Lagrangian $L(g_{kl}, f_{kl}) = L(F, G)\sqrt{-g}$, where g_{kl} and f_{kl} are the metric and electro-magnetic field tensors, and F, G the usual invariants $\frac{1}{2} f_{kl} f^{kl}$ and $\frac{1}{4} f_{kl} f^{*kl}$. As usual the existence of the potential ϕ_k is assumed so that

$$f_{kl} = \frac{\partial \phi_l}{\partial x^k} - \frac{\partial \phi_k}{\partial x^l}$$

giving the identity

$$\frac{\partial \sqrt{-g}\, f^{*kl}}{\partial x^l} = 0 \tag{i}$$

We introduce a second kind of antisymmetrical field tensor p_{kl} by the relation $\sqrt{-g}\, p^{kl} = \partial L / \partial f_{kl}$, and the variation principle for the Lagrangian gives the Eulerian equations.

$$\frac{\partial \sqrt{-g}\, p^{kl}}{\partial x^l} = 0 \tag{ii}$$

(i) and (ii) are the complete set of field equations. The energy-impulse tensor is given by

$$\sqrt{-g} \cdot T_{kl} = -2\partial L / \partial g^{kl} = \sqrt{-g}\,(L g_{kl} - f_{ks}\, p_{lr}\, g^{sr}).$$

It is easy to generalise the action principle in such a way that it contains Einstein's gravitational laws; one has only to add to the action integral the term $\int R\sqrt{-g}\, d\tau$, where R is the scalar of curvature.

Leaving out of account the gravitational laws, the essence of the theory consists in not taking the charge distribution ρ as fundamental but two sets of field strengths **E**, **H** and **D**, **B**, the equations connecting these involving a certain universal constant b (limiting field strength) connected with the classical electronic radius r_0 by $b = e/r_0^2$. For weak fields $\mathbf{E} \equiv \mathbf{D}$, $\mathbf{H} \equiv \mathbf{B}$. By imposing the condition of self-conjugacy on the action function used in the variation principle, Born has shown that these four equal representations of the theory according as (**B**, **E**), (**D**, **H**), (**B**, **D**), or (**E**, **H**) are taken as fundamental. With the third or the **U**-representation in the special form (Born's action function)

$$L = \sqrt{1 + \frac{F}{b^2} - \frac{G^2}{b^4}} - 1$$

the energy-density u is given by

$$4\pi u = b^2\{1 + (\mathbf{D}^2 + \mathbf{B}^2)/b^2 - [\mathbf{DB}]^2/b^4\}^{1/2} - b^2$$

and the total energy and momentum **U** and **G** here form a four-vector, thus successfully exploiting the notion of electro-magnetic origin of mass.

The field equations applied to the special case of an electro-static field with spherical symmetry, and a singularity at the origin, give for the self-energy of the electron the finite value mc^2. Another important case where the field equations have been solved is that of a ring-singularity (40) where also one obtains finite values for the total energy and angular momentum.

In the further development of Born's field theory, the question of the form of the action function has been discussed in great detail. A great number of action functions have been proposed and they all appear to serve the purpose of giving finite energy for the electrical particle equally well. Even the symmetry conditions of self-duality and the conditions of existence of simple algebraic relations between the f_{kl} and p_{kl}-fields do not uniquely specify the action functions. It has been shown in fact (41) that there exist a two-fold infinity of such action-functions.

Coming now to the quantization of the field equations, Born has attempted the same on the lines of the Heisenberg-Pauli theory, taking **D** and **B** as the fundamental variables in place of the potentials ϕ_k. A coherent theory has been built up but its successes must be considered very meagre indeed. The most formidable objection to the quantized theory is the result of Pyce (42) that the electromagnetic angular momentum has its eigen-values as integral numbers thus showing that the theory cannot explain spin. Another purely classical attempt to explain spin of the electron on this theory has met with the same fate (43).

A remarkable coincidence between Born's field theory in its classical aspects, and the purely quantum theory of the positron is worth noticing. In the later theory, for the region $\lambda < \lambda_0$ the field equations can no longer be taken to be linear, and remarkably enough all calculations made hitherto confirm the fact that variations from linearity can be represented by a universal Lagrangian of the same form as that assumed in the Born field theory. The deviation from the superposition-principle is unimportant as long as the field strength remains less than a certain critical value which is remarkably almost equal numerically to the field strength e^2/r_0^2 at the boundary of the electron of radius r_0 on the classical Lorentz-theory. This must have a deep significance, but no satisfactory explanation has so far been advanced.

The most serious objection that can be brought against a theory which postulates an electromagnetic origin of mass is the existence of the neutron, and also the theory of the positron in which negative and positive mass play symmetrical roles. This is not consistent with a theory which does not envisage negative mass even abstractly. Another serious drawback in the non-linear field theory is the absence of a definite principle which serves to specify the action-function to be used in the variation principle.

Rosen (44) has attempted a non-linear field theory by attempting to get rid of singularities in the field. For such a theory he has concluded that the action function L should also contain the potentials ϕ_k besides the f_{kl}, but in order that a gauge-transformation

$$\phi'_\mu = \phi_\mu + \partial\lambda/\partial x_\mu$$

may not change the form of L or the field equations, he has introduced other arbitrary parameters so as to conserve gauge-invariance. By a suitable choice of these arbitrary terms in the form of a scalar function ψ such that it is transformed according to the law

$$\psi' = \psi e^{i\varepsilon}\lambda$$

a consistent theory can be built up in which the non-linear field equations resemble the Gordon-Klein relativistic scalar wave equation. Application to the static spherically symmetric case leads however to a negative mass of the electron.

Another purely classical theory has been attempted recently by Dirac (45) which is not based on any kind of model, retains the Maxwell's equations as they are but tries to remove the infinities by a subtraction formalism of the type used in position theory giving the finite mass as the difference between the infinite negative mass at the centre and the infinite positive mass of the surrounding Coulomb-field. The motion of an electron as disturbed by an electromagnetic impulse is worked out on the basis of the theory and leads to results, of great significance viz., that it is possible for a signal to be transmitted faster than light through the interior of an electron. The finite size of the electron reappears here in a new sense, the interior of the electron being a region of failure, not of the field equations of electro-magnetic theory, but of some of the elementary properties of space-time, although the whole theory is Lorentz invariant. It has been remarked (46) that this theory is not really a solution but an evasion of the problem of explaining the elementary masses since it merely alters the definitions of force, momentum and energy as given in Lorentz's classical theory of the electron. Nevertheless, it might be pointed out that geometrically we have here a very intriguing situation in as much as the interior of the electron is now a region where the geometry and not the field laws, appears to break down.

Related to the problem of self-energy, and the question of space-time in the interior of an electron is a theory developed by A. March (47) in a series of papers on the geometry of a "*Kleinster Raum.*" It is pointed out that quantum mechanics only makes a sparse use of the notion of metric in geometry in as much as it avoids electrons describing definite orbits. Where, however, it actually uses the metric, as for example in the self-energy of an electron or the gravitational energy of a photon, it immediately resolves itself into contradictions. The fundamental notion of March's theory is that the length of the curve-element between two points is to be replaced by

$$\int \left\{ \sqrt{g_{ik} dx^i dx^k} - \gamma \right\}$$

this length being taken in the statistical sense as the result of very many measurements, γ being a small universal positive scalar invariant of the co-ordinates. Thus a statistical metric is introduced which does not permit the use of lengths less than a certain limit γ. The essential nature of the statistical condition is brought out from the fact that, without it, the extension proposed would be merely trivial. Thus if P_1 and P_2 be two points such that the length $P_1P_2 < \gamma$, we can consider a third point P_3 where P_1P_3 and P_2P_3 are $> \gamma$ and define $P_1P_2 = P_1P_3 - P_2P_3$, but a statistical definition would remove this contradiction by showing that P_1 and P_2 cannot be distinguished from each other.

From the mathematical point of view, this geometry would be of quite a complicated type. Thus if P_1, P_2,······P_n be a series of points on a line at distances γ, then $P_1P_n = 0$. This contradicts the axioms of Riemannian geometry, and it means we are here dealing with a non-Riemannian geometry where the metric is defined in the integral and not the differential form. In this geometry we might have, for example, *circles of zero-radius but a finite circumference.* In order to avoid these unnecessary complications, March introduces the notion of "physical points" in place of the mathematical point and like it also extensionless, but with the addition of the statistical concept. Thus all the formal representation of structural natural laws through differential equations are not altered, but the field quantities have a meaning as the average of a large number of measurements. For wave motion, the definitions lose their meaning in the region $\lambda \leqslant \gamma$. If a photon with $\lambda > \gamma$ goes from a frame K to another K′ by a co-ordinate transformation with $\lambda' < \gamma$, the interpretation is that the photon loses its wave character but appears as a static field.

The notion of statistical metric can also be applied to quantum electrodynamics, and March has shown that the Heisenberg-Pauli theory can be developed so as to get rid of all the difficulties connected with infinities, and that too in a relativistic invariant manner. A consequence of the new metric is that in place of Dirac's function an undetermined function $D(x-x')$ appears in the exchange relations. The function D is different from zero inside G (region of dimensions γ), and is transformed on a change of co-ordinates like Dirac's β-matrices. It effects an indeterminacy in orthogonal systems, and hence a breaking off of the spectrum at $\lambda=\gamma$, and the radiation energy will be finite. The proper energy corresponding to terms $i=k$ (which become infinite in the old theory) is here finite and equal to $3e^2/2\gamma$. This expression does not therefore denote any special inner condition on the electron, and the eigen-energy and hence also the mass of the electron are to be considered as magnitudes forced on it by the world geometry. Thus γ and not m is to be considered as a primary natural constant.

The application to the photon gives a finite value for the gravitational energy of the photon which on the old theory leads to infinities as shown by Rosenfeld (48) and Solomon (49). This result has a fundamental significance in the theory because it is here applied to a case where the electron makes no appearance, and shows that γ is not merely a physical constant connected with the electron but an universal natural constant.

Following the ideas of March, Foradori (50) has attempted a revision of the mathematical foundations of the physical space-continuum. The division of the physical continuum into arbitrarily small parts on any fundamental principles makes the geometrical notion of a point, and hence also the "mengentheoretische" point-continuum physically meaningless. These difficulties could be avoided if the space continuum be described not through the connection of unextended points but by the property of divisibility into parts. This is proposed to be done with the aid of the topological "Teiltheorie" which makes continua whose divided parts have a lower limit, understandable mathematically. The basis of this theory is the isomorphy idea, which shows that physical space (represented by its conditioned divisibility) is isomorphous with the geometrical point-continuum (of unlimited divisibility) and this result allows the simple description of physical phenomena by means of a shortest length. The indeterminacy resulting from such smallest volume elements condition a notion of overlapping which leads at once to a statistical metric.

6. Nuclear Physics

A third fundamental difficulty facing quantum mechanics is the theory of the elementary particles—specially the heavy particles which are not dealt with under one comprehensive theory. The specifically nuclear forces have been recently explained with partial success on the basis of the mesotron theory (51) which again meets with difficulties in the region of energies high compared with the rest energy of the mesotron (52). These difficulties have suggested several theories of a geometrical nature.

One of the most striking of these theories is that recently proposed by Born (53) based on what is called the "*principle of reciprocity*". The first idea behind this principle is that, since space-time and energy-momentum appear symmetrically in quantum mechanics, an attempt at the construction of a geometry might be made in the momentum-space just as well as in the co-ordinate space. The second idea is that the geometry is to be of a non-linear type since the non-linearity of the equations of quantum-mechanics seem to survive in all attempts at generalised quantum field theories. The only natural way of introducing non-linearity into a field theory is that of Einstein's theory of gravitation. These ideas have led to the formulation that the domain of the elementary particles has to be considered from the stand point of *momentum space* (since an existence of absolute momentum seems quite plausible) *in which a non-linear geometry with small curvature reigns*, but the curvature equation should contain solely the constants a (of universal length) and b (universal momentum) but not the gravitational constant since gravitation is evidently only a macroscopic effect.

Another purely classical theory to explain the masses of the elementary particles has been proposed by Proca and Goudsmit (54) in which the fundamental idea consists in defining the motion of elementary particles as geodesics of a configuration space carrying besides the ordinary space-time co-ordinates, two others, one corresponding to a charge and the other to spin. Thus the metric of this extended space would be written in the form

$$ds^2 = c^2dt^2 - \sum (dx^i)^2 + \lambda_s \sum (dx^s)^2 + \lambda_e (dx^e)^2$$

and assumed invariant with respect to Lorentz-transformations. This simple classical theory explains the difference in masses of proton and neutron, furnishes a criterion for distinguishing light and heavy particles, completes the relation between electron and photon on Peroni's theory, and finally explains the mass of the mesotron.

Finally mention might be made of a new "Wave-Geometry comprehensively worked out by Japanese theoretical physicists (55) in which they claim to have shown that *Geometry is theoretical physics as it stands*, a claim which, if substantiated, would mean the realisation of the dream of complete geometrisation of physics. The basic assumptions of this theory are the assumption that the domain of existence of physical objects is a 4-dimensional manifold, and that this manifold correspondly to the pure analytical frame (in the topological sense.) The linear operator $ds = \gamma_i dx^i$ is introduced with

$$\tfrac{1}{2}(\gamma_i \gamma_j - \gamma_j \gamma_i) = g_{ij}$$

which is obtained by linearising the metric

$$ds^2 = g_{ij} dx^i dx^j$$

With this *operator* ds is combined the wave function ψ, and the metric of the manifold is taken as $ds\psi$. Such transformations and parallel displacements are considered in the theory as make $ds\psi = 0$ invariant. This theory has been worked out at great length and applied to most problems in theoretical physics, but there appears to be one important trouble in as much as a wave geometry cannot be constructed for the electron-itself (56).

References

(1) BIRKHOFF and NEUMANN: Ann. Math. 37, 823 (1936)
(2) P. JORDAN: Z. Phys. 80, 285 (1933).
NEUMANN and WIGNER: Ann. Math. 35, 29 (1934)
(3) H. WEYL: Group theory and Quantum mechanics.
(4) T. ROSENTHAL: Rev. Mod. Phys. April, 1936.
(5) DELBRUCK and MOLIÈRE: Abh. Preuss. Akad. No. 1, (1936)
(6) MILNE and WHITROW: Z. Astrophys. 15, 263, (1938)
(7) WHITTAKER: Analytical Dynamics, 288 (1927)
(8) SYNGE: Phil. Trans. 226, 31, (1926).
(9) IBID: Ann. Math. 36, 650 (1935).
(10) PRANGE: Ency. Math. Wiss. IV, ii (4), 551 (1935)
(11) BIRKHOFF: Dynamical systems.
(12) HEDLUND: Ann. Math. 35, 787 (1934)
(13) WEYL: Berl. Sitz. May 1918.
(14) O. KLEIN: Z. Phys. 37, 895 (1926).
A. EINSTEIN: Berl. Sitz, 1927, 23-30.
(15) EINSTEIN and MAYER: ibid, 1932, 102.
(16) EINSTEIN and BERGMANN: Ann. Math. 39, 683 (1938)

(17) EINSTEIN. INFELD and HOFFMANN : ibid, p. 65.
(18) STELLMACHER : Math. Ann. 115, 740 (1938).
(19) TOLMAN : Relativity, Thermodynamics, and Cosmology, 1934.
(20) MILNE . Proc. Roy. Soc. 154, 22 ; 158, 324 ; 159, 171 ; 159, 526 ; 160, 165, 313.
(21) ZWICKY : Phys. Rev. 55, 726 (1939).
(22) STERN : Z. Electrochimie, 31, 448 (1925).
(23) DIRAC : Principles of Quantum mechanics (2nd Edn.).
(24) J v. NEUMANN : Math. Grundlagen d. Quantenmechanic (1932), p. 18.
(25) V. FOCK : Phys. Berichte. 10, 2255.
(26) SCHRODINGER : Berl. Ber. (1932), N. 11 12, p. 105.
(27) SCHOUTEN : Journ. Math. Phys. 10, 239 (1931).
(28) BARGMANN : Berl. Ber. (1932) N. 14, p. 346.
(29) INFELD and VAN DER WAERDEN : ibid (1933) N. 1, p. 2.
(30) PAULI and SOLOMON : J. de. Phys. (7), 3, 452, 582 (1932).
(31) SCHOUTEN and VAN DANTZIG : Z. Phys. 78, 639 (1932).
(32) PAULI : Ann. d. Phys. (5), 18, 305 (1933); 337.
(33) DIRAC : Ann. Math. 36, 657 (1935); 37, 429 (1936).
(34) CUNNINGHAM and BATEMAN : Proc. Lond. Math. Soc. 8, p. 7 and 223 resp. (1910).
(35) HEISENBERG and PAULI : Z. Phys. 56, 1 (1929), 59, 169 (1930).
(36) DIRAC : Proc. Roy. Soc. 114, 243 and 270 (1927).
FERMI : Rev. Mod. Phys. 4, 131 (1932).
(37) HEITLER : Quantum theory of Radiation, (1936), 231.
(38) OPPENHEIMER : Phys. Ber. 11, 2510.
(39) M. BORN : Ann. Inst. Poincare, t. 8, fas. 4, p. 155-265.
(40) B. S. MADHAVA RAO : Proc. Ind. Acad. Sci. 4, 355 (1936).
(41) IBID : Proc. Ind. Acad. Sci. 6, 158, (1937).
(42) PRYCE : Proc. Camb. Phil. Soc. 32, 614, (1936).
(43) B. S. MADHAVA RAO : Proc. Ind. Acad. Sci. 6, 129 (1937).
(44) ROSEN : Phys. Rev. 55, 94 (1939).
(45) DIRAC : Proc. Roy. Soc. 167, 148 (1938).
(46) BORN : Proc. Ind. Acad. 8, 310 (1938).
(47) A. MARCH ; Z. Phys. 104, 93, (1937); ibid. p. 161 ; 105, 620 ; 106, 49, 291, 532.
(48) ROSENFELD : ibid, 65, 589 (1930).
(49) SOLOMON : Ibid. II, 162, (1931).
(50) FORADORI : Z. Phys. 161, 537 (1938).
(51) YUKAWA, FROHLICH, HEITLER and KEMMER.
BHABHA—Recent Roy. Soc. Paper on classical theory. (1939).

(52) HEITLER: Phys. Rev. 54, 873 (1938).
(53) BORN: Proc. Roy. Soc. 165, 291; 166, 552 (1938).
Proc. Ind. Acad. Sci, 8, 309 (1938).
(54) PROCA and GOUDSMIT; J. de. Phys. (7), 10, 210 (1939).
(55) HIROSIMA Journal: Wave Geometry: Pamphlets 1-35.
(56) MIMURA and IWATSUKI

ON DEMLO-NUMBERS

BY

V. NARASIMHAMURTI, *Andhra University.*

1. In *The Mathematics Student* Vol. VII, No. 2, some of the properties of Demlo-numbers were considered by D. R. Kaprekar. I shall now prove an interesting result depending upon one of the properties of Demlo-numbers.

In what follows $[x]$ denotes the highest integer in x; $(a)^l$ denotes the digit 'a' repeated 'l' times; and $m_k = \left[\frac{(9)^k}{p}\right] - \left[\frac{(9)^{k-1}}{p}\right]$ where p is an integer. By definition, m_k denotes the numer of all the numbers having k digits and divisible by p.

2. A Demlo number is defined* as an integer which may be partitioned into three groups of which the middle group is of the form $(a)^l$, and the sum of the numbers in the first and the last groups (supposed to have the *same* number of digits) is of the form $(a)^m$; l may even be equal to zero, and the remaining two groups can exist with the above property.

2·1 The following Theorem of D. R. Kaprekar, is used in the sequel:

THEOREM:—*If N is a number with k-digits and P a number consisting of l ($l \geqslant k$) repeated 9's, then $N \times P$ is a Demlo-number in which the first part is $(N-1)$ and the middle part consists of $(l-k)$ repeated nines.*

2·2 It follows that if

$$P=(9)^l,\ N \times (9)^l = (N-1)\ (9)^{l-k}\ \{(9)^k-(N-1)\}.$$

---

*The Mathematics Student, Vol. VI p.

Thus $27583 \times (9)^8 = 2758299972417$

I shall prove, here, the following

THEOREM:—*If a number* $P = N_1\, x_1\, x_2 \cdots x_{l-k}\, N_2$ $(l \geqslant k)$ *of* $l+k$ *digits* (N_1 *and* N_2 *having each* k *digits*) *divided by a number* $p \times (9)^l$ *yields a quotient equal to* $\left[\frac{N_1}{p}\right]$ *with a remainder* $1x_1\, x_2 \ldots\ldots x_{l-k}\, (9)^k$;

then *Case* (i); if p divides neither $(9)^{k-1}$ nor $(9)^k$, the number of such numbers P having $l+k$ digits is precisely m_k;

Case (ii); if p divides both $(9)^{k-1}$ and $(9)^k$ the number of numbers P having $l+k$ digits is precisly m_k;

Case (iii); if p divides $(9)^{k-1}$ but not $(9)^k$ the number of numbers P having $l+k$ digits is (m_k+1);

and *Case* (iv); if p divides $(9)^k$ but not $(9)^{k-1}$ the number of numbers P having $l+k$ digits is (m_k-1);

Proof:—We have

$$\mathrm{N}_1\, x_1\, x_2 \ldots\ldots x_{l-k}\, \mathrm{N}_2 - 1x_1\, x_2 \ldots\ldots x_{l-k}\, (9)^k$$
$$\equiv p \times (9)^l \times \left[\frac{\mathrm{N}_1}{p}\right] = p \times \left[\frac{\mathrm{N}_1}{p}\right] \times (9)^l \qquad \ldots \text{(A)}$$

The left hand side is equal to

$$(\mathrm{N}_1 - 2)\ 9^{l-k}\ (\mathrm{N}_2 + 1), \qquad \ldots \text{(B)}$$

and the right hand side is, by (2·1) equal to

$$\left\{p \times \left[\frac{\mathrm{N}_1}{p}\right] - 1\right\} 9^{l-k'} \left\{(9)^{k'} - p \times \left[\frac{\mathrm{N}_1}{p}\right] + 1\right\} \qquad \ldots \text{(C)}$$

where k' is the number of digits in the number $p \times \left[\frac{\mathrm{N}_1}{p}\right]$.

It follows then that $k' = k$, and so from (B) and (C)

$$\mathrm{N}_1 - 2 = p \times \left[\frac{\mathrm{N}_1}{p}\right] - 1$$

$$\therefore\ \mathrm{N}_1 - 1 = p \times \left[\frac{\mathrm{N}_1}{p}\right]$$

so that $(\mathrm{N}_1 - 1)$ must be a multiple of p.

m_k gives us the number of these $(\mathrm{N}_1 - 1)$'s which have k digits in each. But we want the number of numbers N_1 which have k digits in each. Let us consider the four cases separately;

Case (i): Here all the m_k multiples of p are possible values of $N_1 - 1$. Therefore the number of numbers of the form $P = N_1\, x_1\, x_2\, x_3 \ldots x_{l-k}\, N_2$ is m_k;

Case (ii): Since p is a factor of $(9)^k$, one of the $(N_1 - 1)$'s is equal to $(9)^k$. But this number does not give us an N_1 having k-digits. Therefore one of the m_k numbers has to be rejected. But, since p divides $(9)^{k-1}$, we get an additional $N_1 - 1 = (9)^{k-1}$ which gives a k-digited N_1, and which is not contained in our m_k numbers which are multiples of p. Therefore in this case the number of N_1's is $(m_k - 1 + 1) = m_k$;

Case (iii): Since p divides $(9)^{k-1}$ but not $(9)^l$, from the discussion of the case (ii), we can have the number of N_1's as $m_k + 1$;

Case (iv): A similar discussion, gives the number of N_1's as $m_k - 1$.

3. *Illustration*:—Let us take $l = 5$, and $k = 3$,

Case (i) Let p be equal to 4 which divides neither 99 nor 999.

Here $l - k = 2$. The least $N_1 x_1 x_2 N_2$ is $101\, x_1 x_2\, 899$. The greatest is $997 x_1 x_2 003$. Therefore the total number of such N_1's is

$$\frac{997 - 101}{4} + 1 = 225.$$

By the case (i) of my theorem $m_3 = \left[\frac{999}{4}\right] - \left[\frac{99}{4}\right] = 249 - 24 = 225.$

Case (ii) Let p be equal to 3 which is a factor of both 99 and 999. The least $N_1 x_1 x_2 N_2$ is $100 x_1 x_2 900$, and the greatest is $997 x_1 x_2 003$. So the total number of such N_1's is $\frac{997 - 100}{3} + 1 = 300$. By the case (ii) of my theorem $m_3 = \left[\frac{999}{3}\right] - \left[\frac{99}{3}\right] = 333 - 33 = 300.$

Case (iii) Let p be equal to 11 which is a factor of 99 but not of 999. The least and the greatest $N_1 x_1 x_2 N_2$ are $100 x_1 x_2 900$ and $991 x_1 x_2 009$ respectively. The total number of such N_1's is

$$\frac{991 - 100}{11} + 1 = 82.$$

By my theorem also, it is $= m_3 + 1 = 82$.

Case (iv) Let p be equal to 27 which is a factor of 999 but not of 99.

The least and the greatest $N_1 x_1 x_2 N_2$ are 10999891 and 97399027 respectively. Therefore the total number of such N_1's is

$$\frac{973-109}{27}+1=\mathbf{33}.$$

By my Theorem it is $m_3 - 1 = \mathbf{33}$.

4. The illustration in case (ii) of § 3 gives the solution to the question 1778 of D. R. Kaprekar.*

Note:—$p \times (9)^5 = 3 \times (9)^5 = 299997$ is here a 6-digited number.

A NOTE ON THE HARMONIC ENVELOPE

BY

K. RANGASWAMI, *Annamalainagar*.

1. The envelope of lines cutting two conics S, S′ in point pairs separating each other harmonically is a conic Γ called the harmonic envelope of S and S′. We make use of the following well-known properties of the harmonic envelope viz.,

(1·1) The three conics S, S′ and Γ have a common self polar triangle;

(1·2) The eight tangents of S, S′ at their four common points all touch Γ;

(1·3) More generally, the common tangents of pairs of conics like $S+\lambda S'$, $S-\lambda S'$ in the pencil determined by S and S′ are also tangents to the conic Γ.

In this note† we seek to establish some of the relations between the conics S and S′ when Γ degenerates into a pair of points. We first prove the following main theorem viz.,

THEOREM A:—*The necessary and sufficient condition that the harmonic envelope of two conics S, S′ with four distinct intersections may be a point pair is that the sum of the parameters of two of the line pairs in the pencil* $S+\lambda S'$ *should vanish.*

---

* *The Mathematics Student*, Vol. VII, No. 2.

† Theorems C and D of this note have already been obtained by Mr. Robinson; *Vide* "A Note on the Geometric Interpretation etc..." *Bull. Am. Math. Soc.* vol. *XLI, 1935* pp. 400-402. But theorem A of this note is more fundamental and the synthetic methods used here are more instructive than the verifications in Mr. Robinson's note.

2. Let $S_1 \equiv S - \lambda_1 S'$, $S_2 \equiv S - \lambda_2 S'$, $S_3 \equiv S - \lambda_3 S'$ be the line pairs through the four intersections A, B, C, D of S and S′. Since a line pair as an envelope is the squared point at the intersection of the two lines it follows from (1·3) that the conics $H_1 \equiv S + \lambda_1 S'$, $H_2 \equiv S + \lambda_2 S'$, $H_3 \equiv S + \lambda_3 S'$ have double contact with Γ. In particular when Γ degenerates into a pair of points X, Y by (1·1) X, Y lie on a side of the diagonal triangle of the four points and separate harmonically the diagonal points on that side. Now, if a non-degenerate conic is to have double contact with a conic envelope consisting of a pair of points X, Y it is necessary that the conic should pass through X, Y. As however, only one of the conics H_1, H_2, H_3, say H_1, can pass through XY it is obvious that H_2, H_3 must be degenerate. Thus H_2 is the line pair S_3 and H_3 is the line pair S_2. Hence the parameters of S_2, S_3 form a dyad apolar to the dyad $(0, \infty)$.

From the preceding result we can immediately obtain the relation between the invariants of S, S′ when Γ is a pair of points. In this case the roots of the Lame's equation $\Delta'\lambda^3 + \theta'\lambda^2 + \theta\lambda + \Delta = 0$ are $\lambda_1, \lambda_2, -\lambda_2$; so thar $\lambda_1 = -\frac{\theta'}{\Delta'}$, $-\lambda_2^2 = \frac{\theta}{\Delta'}$ and $-\lambda_1\lambda_2^2 = -\frac{\Delta}{\Delta'}$. Thus the invariant relation[†] is seen to be $\theta\theta' - \Delta\Delta' = 0$.

From theorem A it follows that if S be a non-degenerate conic of a pencil of conics whose line pairs are S_1, S_2, S_3 the conics S′ in the pencil which are so related to S that the harmonic envelope of S, S′ breaks up into a pair of points are the apolar compliments of S with respect to any two of the line pairs. If, however the line pair S_1 be one of the conics related to S in this way the Lame's equation for the system $S_1 + \lambda S$ viz., $\Delta\lambda^3 + \theta\lambda^2 + \theta_1\lambda + \Delta_1 = 0$ will have for roots $0, \lambda, -\lambda$. Thus $\theta = 0$ i.e, S_1 is out-polar to S so that the four points ABCD form a harmonic set on S. We have thus:

THEOREM B:—*In a pencil of conics with four distinct intersections there are three nondegenerate conics so related to a given nondegenerate conic of the pencil that their harmonic envelope breaks up into a pair of points, unless the base points form a harmonic set on the given conic in which case there are only two.*

3. Let the line pairs AB, CD; AC, BD; AD, BC intersect in P, Q, R respectively and let the harmonic envelope of S, S′ consist of the pair of points X, Y on PR. By (1·2) it is easily seen that the tangents at two of the base points say A, C to S and at the other base points C, D to S′ pass through one of the points X, Y—say X. Similarly the tangents at A, C to S′ and at B, D to S pass through Y. Let XA, XB meet CD in V, W respectively. Now, the line pairs X (AC, BD, PP) belong to an involution pencil, i.e., VC, WD, PP form an involution range. Therefore, $VD\cdot WP\cdot PC + CW\cdot DP\cdot PV = 0$ from which we have (VPCD) = −(PWCD)· But (VPCD) = A (ABCD) on S and (PWCD) = B (ABCD) on S′. We have thus the result:

[†] *Vide* CASEY: *Treatise on Analytical Geometry*, page 481, example 4.

THEOREM C:—*A necessary and sufficient condition that the harmonic envelope of two conics S, S′ with four distinct intersections may break up into a pair of points is that the cross-ratio of the four points in some order on one of them be the negative of the corresponding cross-ratio on the other.*

Let the tangents at A to S, S′ meet S′ and S in U and U respectively By Theorem A, S and S′ are apolar compliments of each other with respect to two of the line pairs, say $S_1 \equiv$ (AB, CD), $S_3 \equiv$ (AD, BC). Hence by (1·3) the harmonic envelope of S_1, S_3 which is a conic touching the lines AB, BC, CD, DA belongs to the range determined by S and S′. Therefore by Sturm's Theorem A (UU′, BD) = −1 i.e., (UA, BD) = −1 on S and (U′A, BD) = −1 on S′. Thus since C is a point on S and S′, C (UA, BD) = −1 and C (U′A, BD) = −1 so that CUU′ are collinear. Similar relations hold good for the points B, C, D. Calling U, U′ the associated points of A, we have:

THEOREM D*:—*If two conics with four distinct intersections be such that their harmonic envelope breaks up into a pair of points, the associated points of each intersection are collinear with another intersection.*

---

* *vide* Sommer ville: *Analytical conics*, p. 293, example 35.

NOTES AND DISCUSSIONS

Joule's "Minimum Dissipation" Theorem

In all standard treatises on Electrical theory, it is proved that when an electric current distributes itself in a net-work of conductors in accordance with Kirchoff's laws, the expression $\sum C^2R$ is a minimum, C being the current flowing in a typical conductor of resistance R; i.e. if the same total current were distributed in some other way consistent with Kirchoff's first law (viz. continuity of current), then the value of $\sum C^2R$ in this case would be greater than that for the natural distribution determined by Kirchoff's second law (which is an immediate consequence of Ohm's law. Now since the heat generated in a typical conductor in C^2R per second, the result obtained above is interpreted as follows :—

"In any system of conductors in which there are no internal electromotive forces the heat generated by currents distributed in accordance with Ohm's Law is less than if the currents had been distributed in any other manner consistent with the actual conditions of supply and outflow of the current."

The quotation is from *Maxwell's Electricity and Magnetism,* Volume I, section 284, and all authors that we have been able to consult draw the same conclusion. The result is known as "JOULE'S LAW OF MINIMUM DISSIPATION" and it has been accepted without question for well-nigh a century,—probably on account of its analogy with other well-known "minimum" theorems of Physics. It is the purpose of this note, however, to show that the conclusion quoted above is not justified.

The rate of heat-generation in a conductor to which a current C is supplied is CV, V being the total fall of potential in the conductor. This result is independent of Ohm's Law. If Ohm's Law is true, $V=CR$ where R is a constant for the conductor and then the rate of heat-production is C^2R. If, however, Ohm's Law is not true, the rate of heat-production, CV, is not equal to C^2R; and the minimum property of the expresion $\sum C^2R$ (which is proved) is not equivalent to the minimum property of heat-generation (Joule's Law). In fact it is obvious that so long as the total current supplied to the net-work, at a given potential difference, remains unchanged, the rate of heat-generation is independent of the distribution of current, in the various conductors. In other words Joule's Law is not true.

Royal Institute of Science, *Bombay.* } K. R. GUNJIKAR, and H. P. OZA.

On the number π

(Dedicated to the shades of my pupil Celina Glass)

We give in this note a brief calculation of the number π to two places of decimals, by the method of Archimedes. This calculation can be understood by pupils of secondary schools.

Denoting by a_n a side of a regular n-gon inscribed in a circle with the radius 1, we have

$$a_6 = 1$$

$$a_{2n} = \sqrt{2 - \sqrt{4 - a_n^2}}$$

To prove the last relation, we have the equations

$$a_n = 2 \sin \pi/n, \quad a_{2n} = 2 \sin \pi/2n,$$

since the radius of the circle is unity.

Therefore

$$\begin{aligned} a_{2n} &= 2\,(\sin^2 \pi/2n)^{\frac{1}{2}} \\ &= 2\sqrt{\tfrac{1}{2}(1 - \cos \pi/n)} \\ &= 2\sqrt{\tfrac{1}{2}(1 - \sqrt{1 - \sin^2 \pi/n})} \\ &= 2\sqrt{\tfrac{1}{2}(1 - \sqrt{1 - a_n^2/4})} \\ &= \sqrt{2 - \sqrt{4 - a_n^2}}. \end{aligned}$$

So, we have

$$\begin{aligned} a_{12} &= \sqrt{2 - \sqrt{3}} \\ a_{24} &= \sqrt{2 - \sqrt{2 + \sqrt{3}}} \\ a_{48} &= \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{3}}}} \\ a_{96} &= \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{3}}}}} \qquad \ldots \quad (1) \end{aligned}$$

The side A_{96} of a regular 96-gon circumscribed to this circle we get from formula

$$A_{96} = \frac{a_{96}}{\sqrt{1 - \frac{1}{4} a_{96}^2}}$$

This follows from the fact that

$$\begin{aligned} A_n &= 2 \tan \pi/n \\ &= 2\,\frac{\sin \pi/n}{\cos \pi/n} \\ &= \frac{a_n}{\sqrt{1 - a_n^2/4}}. \end{aligned}$$

From inequalities between the circumferences of these figures

$$96\, a_{96} < 2\pi < 96\, A_{96},$$

we obtain

$$48\, a_{96} < \pi < \frac{48\, a_{96}}{\sqrt{1-\frac{1}{4}\, a_{96}^2}} \qquad \ldots \quad (2)$$

We give now values of the roots necessary for the calculation of a_{96} from formula (1). We calculate the upper bounds of the errors as follows ($A>0$, $\alpha>0$):

$$\sqrt{A+\alpha}-\sqrt{A}=\frac{(\sqrt{A+\alpha}-\sqrt{A})(\sqrt{A+\alpha}+\sqrt{A})}{\sqrt{A+\alpha}+\sqrt{A}}=\frac{\alpha}{\sqrt{A+\alpha}+\sqrt{A}}.$$

Thus

$$\sqrt{A+\alpha}-\sqrt{A}<\frac{\alpha}{2\sqrt{A}}.$$

Values of the roots	Upper bounds of the errors
$\sqrt{3}=1{\cdot}732050\ldots$	0
$2+\sqrt{3}=3{\cdot}732050\ldots$	0
$2+\sqrt{3}=3{\cdot}732050$	10^{-6}
$\sqrt{2+\sqrt{3}}=\sqrt{3{\cdot}732050}=1{\cdot}931851\ldots$	$\frac{10^{-6}}{2\times1{\cdot}9\ldots}<10^{-6}$
$2+\sqrt{2+\sqrt{3}}=3{\cdot}931851\ldots$	10^{-6}
$2+\sqrt{2+\sqrt{3}}=3{\cdot}931851$	$10^{-6}+10^{-6}=2\times10^{-6}$
$\sqrt{2+\sqrt{2+\sqrt{3}}}=\sqrt{3{\cdot}931851}=1{\cdot}982889\ldots$	$\frac{2\times10^{-6}}{2\times1{\cdot}9\ldots}<10^{-6}$
$2+\sqrt{2+\sqrt{2+\sqrt{3}}}=3{\cdot}982889\ldots$	10^{-6}
$2+\sqrt{2+\sqrt{2+\sqrt{3}}}=3{\cdot}982889$	$10^{-6}+10^{-6}=2\times10^{-6}$
$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}=\sqrt{3{\cdot}982889}=1{\cdot}995717\ldots$	$\frac{2\times10^{-6}}{2\times1{\cdot}9\ldots}<10^{-6}$

Values of the roots	Upper bounds of the errors
$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}=1{\cdot}995717$	$10^{-6}+10^{-6}=2\times10^{-6}$
$2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}=0{\cdot}004283$	2×10^{-6} (excess)
$a_{96}^2=2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}=0{\cdot}004281$	2×10^{-6}
$a_{96}=\sqrt{0{\cdot}004281}=0{\cdot}06542\ldots$	$\frac{2\times10^{-6}}{2\times0{\cdot}05}=2\times10^{-5}$
$a_{96}=0{\cdot}06542$	$2\times10^{-5}+10^{-5}=3\times10^{-5}$
$48\ a_{96}=48\times0{\cdot}06542=3{\cdot}14016$	$48\times3\times10^{-5}<200\times10^{-5}$ $=2\times10^{-3}$

Then

$$3{\cdot}14<48\ a_{96}<3{\cdot}143,$$

$$\frac{48\ a_{96}}{\sqrt{1-\frac{1}{4}\ a_{96}^2}}<\frac{3{\cdot}143}{\sqrt{1-\frac{1}{4}\times0{\cdot}012}}=\frac{3{\cdot}143}{\sqrt{1-0{\cdot}003}}=\frac{3{\cdot}143}{\sqrt{0{\cdot}997}}$$

$$=\frac{3{\cdot}143}{0{\cdot}998\ldots}<\frac{3{\cdot}143}{0{\cdot}998}=3{\cdot}14\ldots$$

Therefore we have from inequality (2)

$$3{\cdot}14<\pi<3{\cdot}14\ldots$$

and finally $$\pi=3{\cdot}14$$

Lwòw, Poland. MAURICE SPERLING.

GLEANINGS

The real tragedy about Ramanujan was not his early death. It is of course a disaster that any great man should die young, but a mathematician is often comparatively old at 30 and his death may be less of a catastrophe than it seems. Abel died at 26 and, although he would no doubt have added a great deal more to mathematics, he could hardly have become a greater man. The tragedy of Ramanujan was not that he died young, but that, during his five unfortunate years, his genius was misdirected, side-tracked and to a certain extent distorted.

G. H. HARDY: Lecture on "The Indian Mathematician Ramanujan" *Amer. Math. Monthly*, 1937 p. 142.

REVIEWS

R. WEYRICH *Die Zylinderfunktionen und ihre Anwendungen:* B. G. Teubner, Leipzig, 1937.

This is a small and compact book of about 140 pages proposing to establish the fundamental properties of cylinder functions starting from Sommerfeld's integral representation of these functions. A knowledge of the theory of functions of a complex variable is presupposed, which enables the author to deduce the fundamental properties of these functions quite simply and elegantly.

The book is divided into twelve sections, in the first two of which the author, starting from a general plane wave function, arrives at the spherical and cylindrical wave functions by the so-called *"Uberlegungsprinzip."* Thereby a direct approach to the intuitive grasp of the cylinder functions is made possible. This principle leads us to Sommerfeld's integral representation of the two Hankel's functions of zero order, from which the Hankel's functions $\left(H_n^{\mu}, \mu = 1, 2\right)$ of arbitrary order, the Bessel's functions and differential equations satisfied by them are deduced. In section **3** the author proves periodicity and other properties of these functions together with their representation in power series. Sections **4** to **7** are devoted respectively to asymptotic expansions of Hankel, Debye and Airy; representation by integrals of Poisson's type; the functional equation satisfied by cylinder functions and the zeros of cylinder functions. The next five sections deal with miscellaneous topics. In **8**, we find some integrals involving cylinderfunctions, while 9 is devoted to Lommel's transformation and to the functions of anger, Weber-Lommel, and Sturve. In **10** addition and multiplication theorems are deduced and in **11** the eigenvalve problem of a special type of Sturm-Liouville equation and the expansion of an arbitrary function by means of Bessel functions are studied. The last section deals with some physical application. Some problems in small oscillations, electromagnetic wave theory and heat conduction are considered.

FLORIN VASILESCO—*La Notion de Capacité*—Hermann et Cie Paris, 1937 (Actualités Scientifiques et Industrielles) N° 571.

FLORIN VASILESCO—*La Notion de Point Irrégulier*—Hermann et Cie, Paris, 1938 (Actualités Scientifiques et Industrielles) N 660.

The Dirichlet problem is one of the best illustrations of the apparently paradoxical saying of H. Poincaré that a problem is never solved but only more or less solved, because its so called solution is really a new problem in disguise. Thus Riemann believed that he had solved the Dirichlet problem, but his demonstration, which depends on the minimum on an integral raises the question of the existence of such a minimum, and Weierstrass has shown that the minimum is not always attained. After Riemann, mathematicians of mark like Neumann, Schwarz, Ponicaré, Fredholm did admirable research work to solve the same problem. But they were obliged to consider only *certain* domains. So their results were again setting new problems, in particular the following one: is it always possible to define a solution in the most general case of a domain ?

That the answer must be in the negative was proved first by Prof. H. Lebesgue, in 1913. In the meantime, the theories of point sets, and of functions of real variables, had been so considerably improved that it was possible to tackle the Dirichlet problem from an altogether new point of view. A great number of valuable investigations followed the article of Prof. Lebesgue, and little by little a new theory of the Dirichlet problem was constituted. The two fascicles under review (and the fasc. N° 578 on "Les nouvelles méthodes de la Théorie de Potentiel et le Problème de Dirichlet" by Prof. Ch. de la Vallée Poussin should be added) attempt at a systematic exposition of the results so far obtained in that direction.

The most fundamental notions are that of regular, and of irregular points, and that of point sets of nul capacity. The former two are due to Prof. Lebesgue. A point p of the boundary S of a domain D is said to be regular if for any continuous function $f(p)$ defined on S, the solution $F(P)$ of the Dirichlet problem – generalized if necessary in the sense of N. Wiener – yields lim. $F(P) = f(p)$ when P tends to p in D. If, for at least one function, this does not obtain, then the point is said to be irregular. Necessary and sufficient conditions that a point be regular have been proposed successively by different mathematicians. Some will remain very useful in practice; but they were not very satisfactory until the notion of capacity was introduced. It was first proposed by N. Wiener for closed sets, then extended to open sets, when they are bounded, by Prof. de la Vallée Poussin. O. Frostmann had given recently new generalizations. The notion of electrostatic capacity of a conductor is well known and by a passage to the limit can be easily defined for an enumerable set of points. Its extension to bounded sets, closed and open, raised, on the contrary, considerable difficulties. His generalization of the Dirichlet problem provided N. Wiener with a method of extension, since it led him to extend the notion of potential. But the decisive step was the introduction of a Stieltjès-Lebesgue integral to express without any notion of density the potential of any distribution of mass over a set of points. This has been done by Prof. de la Vallée Poussin who could thus generalize the famous method of balayage (sweeping out method) of Poincaré, to general domains. The notion of function of set is there fundamental. O. Frostmann has shown how unnecessary restrictions were fatal as long as the notion of density was kept, for it is only a secondary notion, while mass and potential are two primitive notions. The methods of these authors are analytic.

Proff. Fekete, Polya and Szegö have proved that exactly as in the case of a sphere, the notion of capacity is one which it is possible to define geometrically. In the case of a closed set, in particular, it will be the transfinite diameter of the set.

This notion of capacity of a set has made it immediately possible to investigate the distribution of the irregular points on the boundary of the domain, and the existence of a limit of the harmonic function on the same boundary. In the theory of integration, a fundamental notion is that of nul sets, i.e. sets of measure equal to zero in the sense of Lebesgue. The theory of the Dirichlet problem calls into play the notion of sets of nul capacity. They are a sub class of sets measurable B (in the sense of Borel) and of measure zero. The boundary of any bounded domain D can be divided into two parts, an "improper" part, always of nul capacity, and its complement, the "reduced" part. A harmonic function bounded in the vicinity of a point of the improper part is harmonic even at these points which are always part or totality of the *interior* boundary. Then the "reduced part" itself can be divided into two parts: a set of nul capacity formed by the irregular points, and the set of the regular points. Both these sets may be dense everywhere. At an irregular point the Green's function does not tend to zero.

Another extremely important application of the notion of capacity has been made to the problem of the uniqueness of a solution of the Dirichlet problem in a general domain. The following theorem has been established by Prof. F. Vasilesco: "It is impossible that two harmonic functions be distinct and bounded in a domain D if their difference tends to zero on the boundary of D except perhaps at the points of an improper set (of nul capacity, thèrefore).

New results have been obtained also by application of the notion of capacity to the theory of analytic functions. Prof. F. Vasilesco points out some, but a more detailed exposition of this subject will be found in the treatise of R. Nevanlinna "Eindeutige Analytische Funktionen" (Springer 1936).

It would be difficult to summarize in a few words each of these two fascicles, because they are already a summary. In the first one the notion of capacity is the object of the first three chapters. The fourth chapter deals with its generalizations due to O. Frostmann, who has considered generalized potentials. A last chapter reviews the principal properties of sets of nul capacity. The second fascicle treats exclusively of the Dirichlet problem. The first two chapters give a short historical sketch of the theory, and introduce the fundamental notions. The three following chapters give an exposition of the results due to the recent investigations of Prof. de la Vallée Poussin, O. Frostmann, F. Vasilesco, D. Kellog, and M. Riesz, in particular the generalization of the sweeping out method of Poincaré. This is a most admirable application of the integral of Stieltjès-Lebesgue. The sixth chapter gives a clear account of the method of M. Brelot who has treated the inverse problem, making use of the properties of subharmonic functions. The last chapter is about analytic functions.

That Prof. F. Vasilesco is one of the best specialists to explain a theory where so many very delicate notions and so many results have to be referred to, is obvious. But to be one of the best specialists in a question does not necessarily mean that one will be clear, chiefly when conciseness is required. Yet as to limpidity, precision and, at the same time, erudition, few tracts of the same series can compete with these two. They are really models of elegant exposition.

C. RACINE.

S. MANDELBROJT "*La Règularisation des Fonctions*" (Actualités Scientifiques et Industrielles, No. 733, Price 11 fr.)

Let $f(z)=\Sigma\, a_n z^n$, $g(z)=\Sigma\, b_n z^n$ be two integral functions. Let $A_n = | a_n |$ $B_n = | b_n |$ and let $A(r)$ and $B(r)$ denote the maximum terms in the sequences $\{A_n r^n\}$ and $\{B_n r^n\}$ for a given r. If $A_n \geq B_n$ for all large values of n it is easily shown that $A(r) \geq B(r)$ for all large values of r. But if $A(r) \geq B(r)$ for all large values of *r it is not true* that $A_n \geq B_n$ for all large n. Suppose we plot the points $(n, \log A_n)$. Let $\{\overline{A}_n\}$ be the sequence such that $(n, \log \overline{A}_n)$ is on the side of the least convex polygon enclosing the points $(n, \log A_n)$. Let $\{\overline{B}_n\}$ denote the points similarly associated with $\{B_n\}$. Then it can be shown that $A(r) = \max\limits_{n \geq 0} \overline{A}_n r^n$, $B(r) = \max\limits_{n \geq 0} \overline{B}_n r^n$ and that $A(r) \geq B(r)$ *for all large r implies that* $A_n \geq \overline{B}_n$ *for all large n*. The effect of replacing $\{A_n\}$ by $\{\overline{A}_n\}$ is, so to say, to smoothen out the irregularities of the sequence $\{A_n\}$. The booklet under review develops a general process by which any function satisfying certain postulates can be replaced by a "regularised" function for which converse theorems of the type mentioned above hold good. The class of functions which can be regularised includes any sequence as a special case. The general method of regularisation is developed in the first chapter while in the second chapter it is shown that converse theorems of the type described above are true for these regularised functions Some applications to sequences in connection with integral functions and the theory of quasi-analytic functions are considered towards the end of the book. The idea of regularising a sequence or a function is important since it has applications to several problems in the theory of functions and the booklet under review provides a general method to effect such a regularisation in any given case.

V. GANAPATHY IYER.

BOOK RECEIVED FOR REVIEW

B. R. SETH: *Two dimensional potential problems connected with rectilinear boundaries,* Lucknow University Studies XIII, 1939, pp. 124.

ANNOUNCEMENTS AND NEWS

Mohammad Khaja Mohiuddin Esq., M.A., Lecturer in mathematics, Osmania University has been elected a member of the Society.

Prof. D. D. Kosambi of the Fergusson College, Poona has been elected a member of the Managing Committee of the Indian Mathematical Society in place of Dr. S. Chowla who has resigned.

We offer our hearty felicitations to Dr. K. S. Krishnan of the Indian Association for the Cultivation of Science, Calcutta, who has been recently elected to a Fellowship of the Royal Society. The late S. Ramanujan was the first Indian to be chosen for this signal distinction (at the age of 30). The other Indians to secure a like honour so far are Dr. J. C. Bose, Sir C. V. Raman, Dr. M. N. Saha and Dr. Birbal Sahni.

MATHEMATICAL REVIEWS.—English reading lovers of mathematics will rejoice over this periodical sponsored by the American Mathematical Society and the Mathematical Association of America, which publishes critical reviews of current periodicals and of books. The earliest periodicals to meet this need were the *Jahrbuch über die Fortschritte der mathematik*, founded in 1871 and the *Revue Semestrielle des publications mathematiques* established about the end of the last century, which became absorbed into the former in 1932. From 1931 the *Zentralblatt fur Mathematik* (Julius Springer) has been offering a somewhat quicker service, but the English reader has been all along left in the cold.

Mathematical Reviews which is printed in large size double column pages appears almost monthly and is at present edited by Prof. J. D. Tamarkin of the Brown University and Prof. O. Neugebauer, who was formerly editor of the *Zentralblatt.* It is priced much below its cost price at $13 per year, with a reduction to $6.5 for members of the sponsoring associations and $3·25 to reviewers. This has been rendered possible by the munificence of the Carnegie Corporation and the Rockfeller Foundation. The Brown University which has perhaps the finest mathematical library in the world is housing the scheme and aiding the editorial work.

An ambitious scheme possible only in the Land of Big Ideas and Long Purses is the micro-film service which is offered to subscribers of The Mathematical Reviews. On request from a subscriber the organizers will send him either micro-film photographs at one 35 mm. film per double page (requiring some reading machine or projector) or photo-prints on sheets 11″ × 14″ readable directly, at 2 cents and 16 cents respectively.

The International Congress of Mathematicians which was planned to meet in Cambridge, Mass, U. S. A. in September 1940 has been "postponed until a more favourable time." All membership fees paid so far have been returned.

The authorities of the Madras University have declared Mr. A. Narasinga Rao qualified for the degree of Doctor of Science. The subject of the thesis was "Studies in Circle Geometry."

SOLUTIONS TO QUESTIONS

Question 1694

(V. THEBAULT, Le Mans, France):—With the figures x, $x+1$, $x+2$, $x+3$, $x+4$ form a number whose square contains the figures 1, 2, 3, 4, 5, 6, 7, 8, 9 each taken once.

Solution by the proposer

Let $n = abcde$ be an integer of 5 digits selected from the 5 numbers x, $x+1$, $x+2$, $x+3$, $x+4$. Since n^2 must be a number between 10^8 and $10^9 - 1$ we have

$$12435 \leqslant n < 31624.$$

Hence the first digit in n must be 1, 2 or 3. So

$$1+2+3+4+5=15 \leqslant a+b+c+d+e \leqslant 3+4+5+6+7=25.$$

Also n^2 is a number with digits 1 to 9 taken once. Hence it is divisible by 9. So n is divisible by 3. Therefore

$$a+b+c+d+e=5x+10=5\ (x+2)$$

is divisible by 3 as well as 5. Hence $a+b+c+d+e=15$. And n is a number whose digits are selected from 1, 2, 3, 4, 5. Considering those beginning with $a=1$, there are 24 numbers of which only the number $n=12543$ for which $n^2=157326849$ satisfies the required condition. Similarly considering those beginning with 2, 3, 4 or 5 we find that none of these satisfy the required condition. Hence $n=12543$ is the only solution.

Also solved by N. K. Narasimhamurty.

Question 1708

(V. THEBAULT, Le Mans, France): With the figures 1, 2, 3, 4, 5, 6, 7, 8, 9 taken each once, form the least and biggest numbers which are perfect squares.

Solution by the proposer

Any number whose square consists of the nine non-repeated digits 1, 2, ..., 9 is of the form $N = abcde$. Since the sum of the digits in N^2 is 45, it follows that N^2 is divisible by 9. So N is divisible by 3. Hence $a+b+c+d+e$ is divisible by 3. It is seen by calculation that $10000 < N < 316263$.

To find the least N, we note that $a=1$ in this case. Next $b=0$ or 1. Now $b \neq 0$ for, if $b=0$, we have $(10cde)^2=(10^4+cde)^2-10^8+2\cdot 10^4\cdot cde+(cde)^2$. If $cde \leqslant 499$, $2\cdot 10^4\cdot cde < 10^7$ and N^2 contains a zero digit. If $N=500$, N^2 contains 4 zero digits and if $N \geqslant 501$, $2\cdot 10^4\cdot cde > 10^7$ and N contains two repeated digits 11 in the first two places. So, $b=1$. Hence the number is of the form $11cde$.

Now, $e \neq 0$ since N^2 will then end in 00. Also $e \neq 1$ or 9 since N^2 will end in 1 and since N^2 begins with 1, it will contain 1 twice. So $e=2, 3, \cdots, 7$ or 8. Using the fact that $1+1+c+d+e$ is divisible by 3, we get that $c+d+e$ must be one of the numbers 4, 7, 10, $\cdots$, 22 or 25. Considering the case $e-2$, we find that $c+d$ must be one of the numbers 2, 5, 8, $\cdots$, 20 or 23. So cd must be of the form 02, 20, 11, 05, 50, 14, 41, 23, 32, 08, 80, etc. It will be seen that none of these are suitable since the square N^2 will contain repeated digits. Similarly the other values of e can be dealt with and it will be found that $cd=82$ is the only suitable combination. So the least N is 11826.

A similar procedure will give that the greatest N is 30384. So the least required number is $N^2=139854276$ and the greatest is $N^2-923187456$.

Question 1713

(FRANCIS REGAN, ST. Louis U. S. A.):—Prove that if through any point V on a conic pairs of lines are drawn making supplementary angles with a fixed straight line in the plane, the envelope of the chords which join the intersections of these pairs of lines with the conic is, in general, a hyperbola.

Remarks by T. K. Raghavachari.

The result contained in the question is not correct.

Taking V for the origin and the lines through V which are parallel and perpendicular to the given line L (say) for axes, the equation of any conic S (not passing through V) may be written as $ax^2+2hxy+by^2+2gx+2fy+c=0$. Now if the line $lx+my+n=0$ meets the conic in P, Q so that VP, VQ make supplementary angles with L, it is easily seen that the line envelopes the conic whose tangential equation is $hn^2-gmn-fnl+clm=0$. In particular when $h=0$ i.e., when one of the axes of the conic is parallel to L, the envelope is a parabola and when $c=0$ i.e., when the conic passes through V, the envelope breaks up into the point V and another point.

Taking the case when V lies on the conic, let the lines through V perpendicular and parallel to L meet the conic S in O, O′ and let T be the pole of OO′ w.r.t. the conic. We notice that:

(i) VO, VO′ are the bisectors of the angles between VT and the tangent at V to the conic.

(ii) When L is parallel to an axis of S, OO′ is a diameter.

(iii) If L turns round a fixed point K, OO′ turns round F the Fregier point of V.

QUESTIONS FOR SOLUTION

1785. (V. THEBAULT, Le Mans, France): Find three numbers A, B, C each of 4 digits such that the 16 digit number AABB may be the square of CC.

[Example: $1976\ 1976\ 8025\ 8025 = 4445\ 4445^2$]

1786. K. S. K. IYENGAR. Given a class $c(f)$ of functions $f(x)$ defined in $0 \leqq x \leqq 2\pi$ with the following properties

(1) f is L^2 in $(0, 2\pi)$

(2) the Fourier series of f converge in the mean to f uniformly in the class $c(f)$.

(3) If f_1 and f_2 are any two functions of $c(f)$ then

$$\int_0^{2\pi} (f_1 - f_2)^2\, dt \geqq k \text{ where } k \text{ is a positive constant;}$$

prove that

(A) (1) the class $c(f)$ is enumerable

(2) If the sequence $\{f_n\} \equiv c(f)$ and $R_n = \left\{\int_0^{2\pi} f_n^2\, dt\right\}^{\frac{1}{2}}$

then the index of convergence of R_n is finite;

(B) If the class c consists of trigonometric polynomials of degree not greater than n_0 and satisfy condition (3) given in the beginning then the index of convergence of the corresponding R_n's is not greater than $2n_0+1$;

(C) A theorem similar to (A) and (B) can be emunciated with regard to algebraic polynomials, the index of convergence being not greater than n_0+1

1787. (M. N. KRISHNASWAMI): If k integrals

$$f_r\,(x_1 \cdot \cdot \cdot \cdot\ x_n\ x) = g_r (r = 1,\ 2,\ \cdot \cdot \cdot\ k)$$

of the system

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \cdot \cdot \cdot \cdot = \frac{dx_n}{X_n} = \frac{dx}{X}$$

are known, show that a last multiplier of the reduced system is given by $M \div \dfrac{\partial(f_1 \cdot \cdot \cdot f_k)}{\partial(x_1 \cdot \cdot \cdot x_k)}$. Examine the case when n integrals of the system are known.

GLEANINGS

Nature of Modern Mathematics

The feature of the modern age which strikes the eye at first sight is its wide and varied use of machinery and the harnessing of natural forces to the service of human needs. That which has made this possible is the mathematical outlook which from the crude beginnings of early human history has now developed into a self-confident full-dress view of the nature of external Reality, founded on the concept of Natural Law.

Philosophical investigations into the nature and method of mathematical science have revealed that Mathematics is essentially a science of relations, and is founded in the last resort on the notion of serial order. The two working concepts of Mathematics—Number and function—are, for instance, purely relational nature, even the 'Numbers,' which are the entities with which Mathematics works, are empty of all concrete content; their being is exhausted in the order relations which define them. In short, Mathematics thinks away the intrinsic essence of the entities with which it deals, there remaining no longer their being-for-itself, but only their relational being or being-for-others. This fact is expressed strikingly in Russell's paradoxical definition, 'Mathematics is the science in which we do not know what we are talking about, nor whether what we say is true.'

The typical being-for-itself is revealed in the notion of the self which arises in self-consciousness. The investigation of the nature of the self which arises in self-consciousness. The investigation of the nature and possibilities of the self through suitable systems of sadhaṇa leads to higher levels of spiritual consciousness and to the experience of God as the upper limit of Being—transcendent and super-relational, or immanent and containing all relations within itself. It follows that the knowledge-effort in man can have two orientations—either towards intrinsic or individual being culminating in its upper limit God, or towards merely relational being or being at its inferior limit in Mathematics. God and Mathematics are thus the extreme poles between which all cognitive experience oscillates; the one typifies the supremely individual being and freedom, the other extrinsic being and mechanism.

R. VAIDYANATHASWAMI

In the Cultural Heritage of India Vol. III p. 402.

THE MATHEMATICS STUDENT

Volume VIII] JUNE 1940 [Number 2

STUDIES IN CIRCLE GEOMETRY*

BY

A. NARASINGA RAO, *Annamalainagar.*

1. The Thesis proper consists of the following papers:—

(i) On Quadratic Equations: *The Mathematics Student*, Vol. IV (1936).

(ii) On the Group of the Miquel-Clifford Configuration: *Annamalai Univ. Jour.* Vol. I (1932).

(iii) On certain Cremona transformations in circle-space connected with the Miquel-Clifford Configuration: *Proc. of the Camb. Phil. Soc.* XXXIII (1937).

(iv) On the Miquel-Clifford Configuration: *Proc. Ind. Acad. Sc.* VII (1938).

(v) The Miquel-Clifford Configuration in the Geometries of Mobius and Laguerre: *Annamalai University Journal*, VII (1938).

(vi) On a principle of duality in circle, sphere and line-Geometries: *Mathematische Zeitschrift* Bd. 44 (1938).

(vii) Studies in Turbine Geometry—I: The concepts of Turbine Geometry *Jour. of the Indian Math. Soc.* III (1938).

(viii) Studies in Turbine Geometry—II: On the sub-geometries of Lie which belong to the Mobius Laguerre pencil: *Proc. Ind. Acad. Sc.* VIII (1938).

(ix) Studies in Turbine Geometry—III: The non oriented line element in two dimensional Mobius Geometry: *Proc. Ind. Acad. Sc.* IX (1939).

(x) Studies in Turbine Geometry—IV: The Topology of oriented and non oriented line elements in the inversive plane: [To appear shortly in the *Proc. Ind. Acad. Sc.*]

---

* [This paper formed the "Introduction" to a thesis with the above title accepted by the Madras University for its Doctor of Science degree. As doctorate theses are not readily available for consultation in India the Editor has reluctantly allowed its publication though he feels that the standard is somewhat higher than that which *The Mathematics Student* has in view.—ED.]

A selection of 13 other published papers representative of other fields in which work was done at different times have been submitted in support of the thesis.

2. One-dimensional Circle-Geometry

The circle in a Euclidean space of one dimension is a pair of points, the centre being their middle point, and the radius half the distance between them. Hence the Mobius Group of circle transformations carrying circles into circles and circles of zero radius into circles of zero radius becomes identical, in this case, with the projective group of the ambient space.

We represent the point pair t_1 t_2 which are roots of $xt^2 - 2yt + z = 0$ by the point $x : y : z$ of a projective plane, the coincident point-pairs (circles of zero radius) corresponding to points on the conic $\Omega : xz - y^2 = 0$. The homographic group of the line now induces on the projective plane collineations which leave Ω invariant. Thus the geometry of point pairs on a line is found to be identical in structure with the geometry of a hyperbolic plane with Ω for its Cayley Absolute, both real and ideal points being taken into account If the line be treated as Euclidean *i.e.* endowed with a parabolic metric defined by a point at infinity ($P\infty$), we have not only the notion of centre and radius but may also define the analogue of the straight line as a point pair of which one member is $P\infty$. The contact of two point pairs would imply the coincidence of one of their points which leads to the somewhat startling result that every circle in 1-dimensional space has only two tangents!

If we differentiate between the orders t_1 t_2 and t_2 t_1 of the point pair, we are dealing with oriented circles in 1-space. The roots being individually known may be subjected to independent linear transformations and this constitute a double-binary domain topologically equivalent to the surface of a ruled quadric, the analytical mechanism for effecting this correspondence being provided by a redundant co-ordinate. These topics and the adjunction of ideal elements to secure closure are discussed in the first part of paper (i).

The geometry of point pairs is of interest because it stands at the junction of two important mathematical disciplines when we pass on to spaces of higher dimensions according as we emphasise the binary or the quadratic character of the binary quadratic. One road —the one we are not pursuing—leads to

3. The Geometry of the Binary n-ic

under the group of homographic transformations. The algebraic part of this theory dates from the time of Cayley and Sylvester and was the subject of considerable activity in the hands of Clebsch, Gordon, Aronhold and others. On the geometric side the invariant theory of the binary n-ic is found to be identical—in the sense of Klein—with the geometry of a projective n-space under the group of collineations which carry a particular rational norm curve into itself. Not only has the geometrical picture been used (by Meyer, Stephanos and White to mention a few names) for obtaining a representation with respect to the norm curve of invariants, covariants and of linear systems of binary forms, but it has also been used the other way by Comessatti to work out a geometrical theory of binary forms. A series of striking results by Vaidyanathaswamy and Ramamurti on the study of correlations in relation to the norm curve and its connection with the theory of double-binary equiforms and of quadrics and simplexes related to the norm curve bear witness to the richness of a field which is yet to receive adequate attention.

The study of oriented n-ics appears to have received no attention at all. Orientation implies that we distinguish between certain permutations of the n roots $t_1\ t_2 \ldots\ t_n$ but regard others as equivalent: hence the number of possible orientations is equal to the number of sub-groups of the permutation group on n symbols. In the later part of paper (i) I have tried to indicate roughly some features of geometry of the binary cubic both oriented and non-oriented.

The other road—the one we are to pursue—leads from the Euclidean 2-point to the geometry of circles and spheres in n-dimensional Euclidean Space, and we begin with

4. Circle Geometry in 2 dimensions.

Because of the ease with which it can be drawn, its perfect symmetry and its many applications even in a primitive society, the circle must have been a familiar geometrical object from the very infancy of the human race. The Greeks who were lovers of the beautiful and of the abstract and to whom the world owes the logical development of geometry from simple premises, discovered the well-known properties of the circle which have been codified in Euclid's *Elements*. So completely had the circle captured the hearts of Aristotle, Hipparchus and other astronomers right up to the time of Kepler that they could not conceive of the Creator being so unaesthetic as to use figures other than the circle in his plan of the Universe! The study of sections of a cone, again a product of the genius of the Greeks, assigned the circle to the family of conic sections—a position which gained greater significance with the advent of Cartesian Geometry in the 17th century. The rapid development of projective geometry in the 19th century after a long interval of neglect and the introduction of the circular points at infinity and of its 3-dimensional analogue the imaginary circle at infinity, by Poncelet, made clear the nature of the specialisation of the conic and of the quadric which results in the circle and the sphere. In 1853 young Laguerre a pupil of Chasles, then 19 years old and a student at the Ecole Polytechnique, gave a projective definition of the Euclidean angle to be followed six years later by Cayley's brilliant generalisation expressing all metrical magnitudes as projective relations of the figure to an "absolute" conic or quadric in such a way that when the latter degenerated into Poncelet's point-pair or imaginary circle at infinity the magnitudes were identical with those of Euclidean geometry. It thus became possible to treat circles in a plane by purely projective methods as conics through two fixed points or as stereographic projections of sections of a quadric from a fixed point thereon—the view adopted in Baker's monumental "*Principles of Geometry*".

5. The ideas of Plücker and Klein

Of the rich harvest of ideas which are the legacy of the 19th century the two which concern us most are firstly, the development of the Group concept and its use as a basis for the classification of geometries by Felix Klein, and secondly Plücker's idea that space may be considered not only as an aggregate of points but also of geometrical objects of any kind, the dimension of the space being dependent on the choice of the basic element.

Klein's definition of a geometry as the study of those properties of geometrical objects which are unaltered by the transformations of a particular group given in his Erlanger Program of 1872 not only enabled Euclidean and non-euclidean geometries to be imbedded into projective geometry as sub-geometries thereof but threw fresh light on the significance of Cayley's Absolute in classical non-euclidean geometry. The ideas of the Erlanger Program have been the inspiration behind a large amount of geometrical researches for nearly half a century particularly by way of the creation of new geometries and of the interpretation of Elementar-mathematik vom hoheren Standpunkt aus. The interpretation of the configurations of one geometry from the standpoint of another with a different transformation group is one of the most fascinating fields opened up by Klein, a region in which little work has been done yet.* It must, however, be stated that there are important geometrical schemes which cannot be squeezed into the Erlanger framework and that a more comprehensive basis of classification is being studied under which geometry becomes a study of a "geometrical object" with components. †

Plücker's representation of lines in a 3-space by points on a quadric in 5-dimensions has given a new and modern meaning to the term "circle geometry", which no longer means the investigation of the geometrical properties of one or more circles, but a study of the structural peculiarities of the 3-dimensional manifold of circles in a plane. The structure imposed on the manifold does not arise merely out of the fact that our basic element is the circle but will be determinate only when the transformation group has been specified, that is, only when we have decided what differences we shall ignore and what differences we shall recognise. Our vision now comprehends the totality of all circles in a plane, these being represented by points in a manifold in higher space and we can see in the structure of the manifold the imprint left by our transformation group. Among the transformation groups of circle geometry which have been so far studied are those associated with the names of Möbius, Laguerre and Lie.

6. Möbius Geometry

August Ferdinand Möbius was the most gifted of Gauss' pupils and his name is associated with several concepts which are recognised to be of first rate importance today, for example homogeneous coordinates, the one-sided Möbius band, and Möbius Null-system to name only a few. Klein refers to him as the "eigentlichen Vorlaufer des Erlanger Programms." Möbius deals with non-oriented circles, and the Möbius Group comprises all transformations of circles into circles in which point circles go over into point circles. The inspiration for this study must have come from his master's introduction of the complex variable $z = x + iy$ into analysis; for one of the simplest methods of obtaining the group in question is to represent points in the Gauss plane by means of the complex variable and subject them to the homographic transformations $z' = (az+b)/(cz+d)$ or $z' = (a\bar{z}+b)/(c\bar{z}+d)$. The real transformations form a mixed 6 parameter group consisting of two continuous systems of transformations. From our point of view the group may be defined as the result of inversion in one or more circles, one system corresponding to an even number of inversions and the other to an odd number, the former constituting a sub-group of the Möbius group. To secure (1, 1) correspondence it is necessary to adjoin an ideal "point at infinity" to the Euclidean plane and, if imaginary elements are also taken into account, two imaginary ideal lines through it. The inversive plane may thus be obtained from the projective plane by contracting, as it were, the line at infinity into a point and drawing out the circular points into lines through this point. The geometrical mechanism for effecting this change of closure is stereographic projection and accordingly we find that the inversive group may be represented by projective transformations in 3-space which leave a sphere invariant. In this representation each circle in the plane is represented by the plane whose section of the sphere projects into the circle in question, or if we prefer it, by its pole with respect to the sphere. In this imbedding of inversive geometry of the plane into the projective geometry of space, the two systems of transformations correspond to collineations which carry each of the two systems of generators of the sphere into itself or interchange the two systems. It is also clear that the inversive group of the plane may be identified with the transformation group of the projective plane treated as a double binary domain, by means of isotropic coordinates, when the binary variables are treated as digredient.

---

* A very interesting example of such a study that I have come across is that of the isogonal transformation associated with a triangle in a Euclidean plane reinterpreted in terms of the ideas of Möbius-geometry: Vaidyanathaswamy: A memoir on cubic transformations associated with a desmic system, *Jour. Ind. Math. Soc.* Vol. 17 Suppl.

† Schouten and Haantjes: on the Theory of the Geometric object, *Proc. Lond. Math. Soc.* Vol 42.

Möbius geometry in Euclidean spaces of higher dimensions exhibits practically all the features of the 2-dimensional case except that since a quadric in n-dimensions has two systems of generating regions only when n is odd, the distinctions between the two types of transformations will arise only for inversive geometry in an even number of dimensions. In fact we shall meet with examples later which go to show that circle geometry is highly sensitive to the residue modulo 2 of the dimensional number.

7. The Geometries of Laguerre and Lie

To Edmond Laguerre the French mathematician who has been already mentioned in connection with the projective definition of an angle, we owe another fruitful idea—that of the oriented line (semi-droit) and of the oriented circle. In a series of papers under the name "geometrie de direction" beginning from 1879 he developed this idea and discovered an analogue for Möbius inversion in his "transformation par directions reciproques", or Laguerre-inversion as it is called today. He discovered the invariants under Laguerre inversion of the figure formed by 4 oriented lines, just as Möbius had discussed the geometrical significance of the cross ratio of 4 complex numbers in the Gauss plane. In fact Laguerre geometry may be considered as the dual of Möbius geometry with the oriented line as the basic element, the transformations defining the geometry in Klein's sense being those which carry oriented lines into oriented lines and oriented circles as proper envelopes of oriented lines into oriented circles.

The effect of orienting the circles is to cover the 3-dimensional projective space of Möbius twice over, so that the manifold of all oriented circles becomes one whose fundamental group is the identity, that is, a 3-sphere. An actual representation of oriented circles in a plane by points on a quadric in 4-space, and more generally of oriented spheres in a euclidean n-space by points on a quadric in $[n+2]$ is obtained by the introduction of the signed radius as an additional coordinate, opposite signs corresponding to opposite orientations. We have then $n+3$ homogeneous coordinates connected by a quadratic identity and hence a representation on the *Lie Quadric* Ω of a projective $[n+2]$. The Laguerre group now turns out to be the group of projective transformations in $[n+2]$ which carry Ω into itself, and keep a particular point I thereof called the "point sphere at infinity" invariant. (vide paper vii p. 100).

There is a close parallelism between the Möbius and Laguerre sphere geometries in a Euclidean $[n]$ since both contain $(n+1)(n+2)/2$ parameters. In two dimensions we may exhibit this parallelism by the introduction of the "dual numbers" which appear to have been first used by Clifford but have become familiar to us by the work of Scheffers and Study. These are numbers of the form $a+\varepsilon b$ and are added and multiplied in the usual way except that we take $\varepsilon^2=0$. We may introduce as the coordinates of the oriented line $x \cos \alpha + y \sin \alpha = p$, the dual number $\alpha+\varepsilon p$ so that in

Mobius Geometry	**Laguerre Geometry**
the point (x, y) is pictured by the complex number $x+iy=z$	the line (α, p) is pictured by the dual number $\alpha+\epsilon p=u$

The transformations of the two groups may be represented by

$z'=(az+b)/(cz+d)$ where a, b, c, d are *complex numbers.*	$u'=(au+b)/(cu+d)$ where a, b, c, d, are *dual numbers.*

These transformations may be considered as contact transformations which carry

non-oriented line elements through the same point	oriented line elements on the same oriented line

into other line elements of the same kind in such a way that

the angle between two line elements through a common point	the distance between two line[†] elements on the same oriented line

remains invariant. The general

conformal group	equilong group

of the plane may be represented by the transformation

$z'=f(z)$	$u'=f(u)$

at all places where f is an analytic function.

However, the analogy breaks down at a certain point since, when represented as a sub-group of the projective group in $[n+2]$ with an invariant quadric Ω, the first carries a proper section of Ω into itself while the latter carries a tangential section of Ω into itself. This difference shows itself, for example, in the existence of a system of similarly oriented parallel lines which are all determinate when one of them is given, a situation for which there is no analogue in Möbius Geometry.

With the Norwegian Mathematician Sophus Lie we come nearer to our own times for many of the investigations initiated by him such as his work on continuous groups, on differential equations and on contact transformations have considerable interest at the present day. He was in close contact with two of the greatest geometers of his day—Felix Klein and Gaston Darboux—and to him we owe the study of transformations of oriented circles[*] or spheres into oriented circles or spheres so as to conserve proper contact. In other words the Lie group of Höhere Kugelgeometrie may be considered as consisting of transformations of oriented surface elements in such a way that spheres are carried over into spheres. Since oriented spheres in $[n]$ correspond to the points on a quadric Ω in $[n+2]$, this group is isomorphic with the automorphic group of collineations of Ω in a projective $[n+2]$. Möbius geometry corresponds to the subgroup which keeps a particular point m and therefore also its polar prime w.r.t. Ω invariant, while Laguerre geometry may also be imbedded in Lie's as the sub-group which keeps another point l invariant, this point lying on the quadric (vide paper (vii) pp. 98-101). I have considered in paper (viii) a pencil of geometries obtained by taking as the invariant prime any member of the pencil formed by the 2 primes which are respectively invariant in the Mobius and Laguerre geometries. The points in $[n+2]$ other than those on Ω have had no geometrical significance attached to them. In my last 4 papers I have interpreted them as "turbines" and tried to study their mutual relations and to use them as tools in our study of Sphere geometry.

[†] This is the "engere" Laguerre Group. The wider group will change all tangential segments in the same ratio.

[*] The oriented straight line and plane have no special status in Lie's geometry but are merely circles or spheres of infinite radius.

One of the most interesting correspondences which Lie worked out is one between oriented spheres in [3] and lines in [3] so that spheres which touch correspond to intersecting lines. The introduction of the turbine concept gives this a richer significance as one in which a linear complex corresponds to a turbine and two complexes in involution to "conjugate turbines". Special linear complexes (straight lines) correspond to special turbines (oriented spheres).

8. The Miquel Clifford Configuration

which is the subject of study in papers (ii), (iii), (iv), (v) belongs to Möbius geometry. Three lines (1), (2), (3) have 3 intersections (12), (23), (31) which lie on the circumcircle (123). With 4 lines we have 4 such circles which Wallace in 1804 and Steiner in 1827 proved to be concurrent. The result may have been known earlier since it follows from the property discovered by Lambert in 1761 that the circumcircle of a triangle formed by 3 tangents to a parabola passes through the focus. In 1836 Miquel proved that the points of concurrence of the 5 quadrilaterals derived from 5 lines are concyclic and finally Clifford in 1870 showed that the alternation of concurrent circles and concyclic points persists indefinitely. Meanwhile Miquel showed that his own theorem concerning 5 lines remains true if the lines are replaced by concurrent circles, a result obviously deducible by inversion; but Miquel was unaware of the inversive transformation though Plucker had used it in 1831 in *Crelle's Journal* (Bd. 11), and Miquel's proof deals directly with circles.

As far as I am aware it appears to be G. H. Grace and later E. H. Neville that first inverted the Miquel Clifford configuration and showed that the resulting configuration gained in symmetry, for it now consisted of 2^{n-1} circles and 2^{n-1} points such that n circles pass through each point and n points lie on each circle. The configuration is homogeneous in the sense that each of the points of multiple incidence by which the configuration is characterised stands on the same footing and the figure may be generated from the n circles concurrent thereat. These symmetries are lost in Clifford's original configuration of points lines and circles, thus showing how a configuration suffers when transplanted into foreign geometry.

In his paper referred to above † Grace showed that the Miquel Clifford configuration has no analogue in [3] but has one in [4], while there are configurations in [3] which have no analogue in the plane — another example to showing how sensitive circle geometry is to the residue modulo 2 of the dimensional number.

In paper (ii) I have considered the group of permutations of the points and circles in the M. C. C. (Miquel Clifford configuration) which preserves all the incidence relations. There are two such automorphic groups according as we allow points to be replaced by points and circles by circles or place no restriction whatever. Both the groups are shown to be Abelian groups generated by permutable operations each of period 2.

If the number of circles n concurrent at O and generating the M. C. C. is even there is an associated Miquel Point M which is the last to be obtained in the series. An additional circle through O gives rise to an additional circle through M namely the Clifford circle of the $n+1$ circles through O. Paper (ii) studies the transformation in circle space which carries the one into the other. It is shown to be a Cremona Transformation of the De Jonquiere type whose F-elements of unit multiplicity are the given circles and the element of multiplicity $n/2$ the point circle at O. Clifford n-fold parabolas are shown to give an alternative approach leading to the same result.

---

† Vide a remarkable paper by J. H. Grace on "Circles, Spheres and Linear Complexes", *Proc. Camb. Phil. Soc.* Vol. XVI. For an interesting extension of the M. C. C to n-space and a proof thereby, vide F. P. White: *Proc. Camb. Phil. Soc.* Vol. 22 pp. 684-7.

In paper (iii) the inversive and projective view points are combined to prove Ramaswami Aiyar's result regarding the case in which the Clifford Circle of $2n+1$ lines is a straight line, and to generalise it to the case when the Clifford circle is one of constant radius. Lastly it is shown that the angles at which the circles cut at a point of multiple incidence are the same as those at which they cut at any other point of multiple incidence of the M. C. C. This is interesting as providing a generalisation of the well known theorem in elementary geometry that the angle between a tangent to a circle and a chord through the point of contact is equal to the angle in the opposite segment, which is the form taken by the result when $n=3$.

There is an analogous configuration in Laguerre geometry to the M. C. C. oriented lines and circles taking the place of points and circles and contact replacing incidence. Paper V proves the angle property of the M. C. C. in Möbius geometry obtained in the previous paper by elementary methods and proves also the analogous property which the result given above namely that in M. C. C. of Laguerre geometry the tangential segments marked out on any of the 2^{n-1} oriented lines of the configuration by all the n oriented circles which touch it are the same as on any other of the oriented lines.

PAPER VI

The range and power of Mathematics arises from her ability to isolate and deal with the essential features of a problem ignoring unessential details, and hence it arises that the same mathematical discipline is capable of dealing with several apparently diverse situations. It would be, therefore, more economical to study the central mathematical core in general terms rather than deal with particular realisations thereof. One such investigation is undertaken in paper (vi) where the basic discipline is the invariant theory of projective transformations in $[n]$ which leave a particular quadric Ω invariant. Particular realisations of this are:—

	points on Ω corresponding to
(1) Non-euclidean Geometry	points at infinity;
(2) Möbius geometry in $[n-1]$	points-spheres
(3) Turbine transformations in Lie's geometry in $[n-2]$	oriented spheres

(4) $n=5$ Projective line-geometry in $[3]$	straight lines (special linear complexes)
(5) Study's geometry of rigid displacements in $[3]$ $n=7$	Soma and pseudo-soma

In all these, the conjugacy of a pair of points in relation to Ω represents an important invariant relation. In paper (vi) I study certain linear transformations applied to sets of mutually conjugate elements which will conserve their conjugacy relations and give applications of these results to circle, sphere and line-geometries. Among others, certain interesting generalisations of the Miquel-Clifford and Hart systems are obtained.

The remaining papers deal with

9. Turbine Geometry.

The 'Turbine concept has been introduced into sphere geometry with a view to people the empty spaces in the representative $[n+2]$ where, so far, only points on Ω have been interpreted (as oriented spheres). The turbine is obtained by associating a non-oriented sphere C or radius R with an oriented sphere c of radius r in concentric position, and has thus an additional degree of freedom over the oriented sphere. It may be defined as an aggregate of oriented surface elements whose point locus is C and whose plane envelope is c. In fact we take every surface element † (oriented) of c and slide it in all directions in the tangent plane through the same distance t, the totality of surface elements thus obtained being the turbine. This is the inner specification (c, t) of the turbine through its inner sphere c and tangential segment t. The points of the elements lie on a sphere C of radius R given by $R^2=r^2+t^2$ and since R is thereby seen to be capable of a double sign, we regard C as oriented both ways which is my definition of "non-oriented". The elements cut C at a constant angle α where $\cos\alpha=r/R$. Now two primes cut at an angle which is determinate mod. 2π when they are both oriented and determinate only mod. π when one or both are non-oriented. Since the elements of C are oriented both ways the angle α is determinate mod. π and may be replaced by $\alpha+\pi$. In fact since $\cos\alpha=r/R$, we should take α if R has one sign and $\alpha+\pi$ if R takes the opposite sign, so that $R\cos\alpha$ which enters into the outer co ordinates of a turbine (vide paper vii p. 99) is quite determinate. We thus have an alternative "outer specification" of the turbine through (C, α) in which each of the 2 possible signs for R is associated with a particular value out of α or $\alpha+\pi$. The significance of the non-oriented character of C lies precisely in this freedom to orient C either way, adjusting α accordingly. These considerations must lead us to expect in our formulae r and t^2 (since the sliding is done in all directions) or R^2 and $\cos^2 \alpha$.

---

† By a surface element is meant in hyperspace the association of a point and a prime incident therewith.

When the inner sphere r is fixed and t allowed to vary we have an "inner pencil of turbines" which correspond to a line in $[n+2]$ through the point l, the invariant point of Laguerre geometry. If, on the other hand, C is fixed and α allowed to vary we have an "outer pencil" represented by a line through m, the invariant point of Mobius geometry, the 2 intersections of the line with Ω corresponding to C oriented different ways. In imbedding the concepts and results of Lie geometry and its sub-geometries into turbine geometry, the oriented sphere corresponds to the turbine whose inner and outer spheres coincide. Laguerre and Möbius geometries are specialised from Lie geometry by the fact that they carry inner pencils into inner pencils and outer pencils of turbines into outer pencils respectively. Euclidean geometry which is the intersection of the Laguerre and Möbius geometries is thus the sub-geometry of Lie which carries every outer pencil into an outer pencil and every inner pencil into an inner pencil. The non-oriented sphere of Mobius geometry may be imbedded into turbine geometry by making it correspond to a turbine whose inner sphere is of zero radius. The contact of two turbines is so defined that it corresponds to points whose join touches Ω (vide. postscript to paper vii)

The turbine concept was introduced in a tentative way in paper (vi) by means of "weighted spheres" and without a full idea of their geometric significance. The word itself is due to Kasner who defines it in two dimensions as the configuration obtained by turning the line elements of an oriented circle *the same way round* through the same angle. This has no analogue in higher dimensions and is called by me a "semi-turbine". My 2-dimensional turbine is obtained by rotating the line elements in *both senses* through the same angle and has thus 2 line elements through every point. The main features of the Kasner-turbine and the turbine in 3 dimensions are indicated in Blaschke's, *Vorlesungen uber Differentialgeometrie* Bd. III (1929) S. 51 and S. 231 as "linearer Komplex von Kugeln" but the subject is not pursued further nor are the concepts sharpened for further use. In paper (vii) the foundations are laid for what promises to be an interesting extension of circle and sphere geometries in which the conformal and equilong patterns will intermingle with each other.

The first superstructure to be erected on these foundations is the creation of a pencil of geometries G_k bridging the span between Möbius and Laguerre geometries. The group which defines the geometry G_k is the group of collineations which carry Ω into itself and leave invariant a definite point K on the line ml joining the invariant points m and l of the Möbius and Laguerre geometries; or geometrically, the group of turbine transformations which carry oriented spheres into oriented spheres and turbines of inner radius k into others of inner radius k. By associating with each non-oriented sphere an inner oriented sphere of radius k, a G_k-geometry of non-oriented spheres is worked out which is the analogue of Möbius geometry. Spheres of radius k here play the same role as points in Möbius geometry, and there is a G_k inversion corresponding to Möbius inversion. The incidence of a point and a sphere is now replaced by the "bisection" of a k-sphere by a sphere of radius other than k while corresponding to the angle in Möbius geometry and the tangential segment in Laguerre geometry we have a basic invariant which partakes of the character of both. The Miquel-Clifford and Hart configurations are then generalised by transplantation into G_k-geometry. All these matters are discussed in (viii).

Since turbine geometry is essentially a geometry of suface elements, it seemed desirable to study the inversive plane as an aggregate of line elements—a study undertaken in paper (ix). The non-oriented line elements in the inversive plane are represented homeomorphically on a V_3^4 in a projective [5] which is the join of 2 conics whose planes are skew to each other. The inversive group induces in [5] projective transformations which have V_3^4 as an invariant variety. The structure of V_3^4 is studied and the loci thereon examined. Thus corresponding to line elements whose points lie on a circle, we have on V_3^4 a rational normal ruled surface (V_2^4) and on this we have "leitkurven" of lowest order 2 which correspond to turbines with the given circle for outer circle. "Leitkurven" of orders 3 and 4 are also interpreted. By projection from one of its generating lines on to a [3], we get a representation of line elements of the inversive plane by points in a [3]. The group induced in this [3] by the inversive group of the plane is a cremona cubocubic group some of whose features are studied. The configurations in [3] corresponding to the line elements of a Hart System and of the Miquel Clifford Configuration are obtained. It is interesting to note in this connection that if a fixed point is kept invariant on the inversive plane (say the point at infinity) thus making the geometry euclidean, the group in [3] becomes the projective group which keeps invariant a line in [3], two points on it and two planes through it, a group whose geometry which has been studied both in its algebraic and differential aspects by Blaschke in his recent book *Ebene Kinematik* (Hamburger Math. Einzelschriften, Heft 25, 1938) under the name "Quasielliptische Geometrie". Quasielliptic space is no other than the parameter-space of the group of movements in the Euclidean plane and its group is the direct product of the group of movements with itself.

The last part of paper (ix) indicates some of the features of the analogue of line element geometry in the geometry G_k. It is found to be the geometry of "rods" (point pairs at a constant distance $2k$) under the group of G_k inversions.

In the last paper (x), I have studied the topology of the totality of line elements, both oriented and non-oriented in the inversive plane. The connection between the group of a geometry and the topology of the operand-manifold arises out of the need for a suitable closure of the latter so as to secure that each transformation of the geometry shall effect a (1,1) correspondence of the elements with themselves. It is shown that the oriented line elements of the inversive plane are topologically equivalent to the points of a projective 3-space,—a known result, since the manifold is homeomorphic with the oriented line elements of a 2-sphere. When the elements are not oriented the Poincare group is a cyclic group of order 4 and the variety is homeomorphic with a lenticular space in which points on each quadrant of the circular edge are considered identical with those obtained by a rotation about the axis through a right angle. I have not seen a specific statement of this result, though it almost follows by piecing together the known results in the topology of 3-dimensional manifolds.

Paper (x) concludes the thesis. Where known results have been used they are acknowledged in the body of the paper where they are quoted. All the other results are believed to be new.

Before concluding I may perhaps be allowed to indicate the directions in which further work may be carried on in which the ideas and results of turbine geometry will prove a valuable tool.

10. Generalisations and extensions of the Hart Configuration

The Hart Configuration consists of two tetrads of circles of which each circle of one tetrad touches all the circles of the other tetrad. This configuration belongs to Möbius geometry and has been the subject of elaborate study by many writers, particularly Coolidge (Treatise on the circle and the sphere), Study (*Mathematische Annalen* Bd. 49) and Baker (Principles of Geometry Vol. IV). That it will still repay study may be granted if it is pointed out that there are newly discovered facts such as the result by Dr. Vaidyanathaswamy that all the 8 circles of a Hart system belong to a cubic one-parameter family of circles which do not fit into the existing scheme, that nothing seems to be known about the existence of similar configurations in higher space, and that even in two dimensions the general features of two tetrads of circles which cut each other at the same angle do not appear to have been studied. Again, there should be analogous configurations in Laguerre geometry in which the length of the tangential segments between spheres one from each system is the same. The study of the Hart Configuration by means of oriented circles and spheres should prove somewhat easier since the high degree of irrationality involved in the Problem of Appolonius (as evidenced by the generous distribution of the $\sqrt{}$ sign in almost every page of Study's paper) gets reduced when oriented circles are used. Thus, touching 3 oriented circles there are only 2 oriented circles instead of 8 as in the case of non-oriented circles.

The existence of Hart pentads of spheres in [3] seems to follow from Schläfli's double-six of lines. For, the double-six corresponds on Plücker's Quadric in [5] to an in-circum simplex $A_1 \ldots A_6$ whose faces touch the quadric at say $\alpha_1 \ldots \alpha_6$. In other words we have two sets of points A_r, α_r $(r=1, \ldots 6)$ such that $A_r \alpha_s$ is a generator if $r \neq s$ and not a generator when $r=s$. If we identify* Plucker's quadric with the Lie quadric Ω in [5] which images the oriented spheres in [3], we have 2 sets of 6 spheres of which each has proper contact with all the spheres of the other set except the corresponding member. Suppose now, that we can find a point O not on Ω such that OA_1, $OA_2 \ldots OA_5$ meet the opposite primes in points lying on Ω. Then OA_1 meets a generator through α_1, OA_2 meets a generator through α_2, and so on up to OA_5. We now subject the whole figure to a collineation which carries Ω into itself and O into the point m which is the invariant point of Möbius geometry (vide paper vii p. 100), a point which, it will be remembered, does not lie on Ω. Denote the points after transformation by the same letters as before. The point sets $A_1 \ldots A_5$ and $\alpha_1 \ldots \alpha_5$ correspond to two pentads of spheres such that each of one set touches properly all the non-corresponding members of the other set and has improper contact with the corresponding member; for sphere A_1 with its orientation reversed is represented by the intersection of mA_1 with Ω and since this lies on a generator through α_1 it follows that A_1 reversed touches α_1. If the orientations are ignored we have a double-pentad of Hart spheres in Möbius Geometry.

---

* They do not have the same signature, so that the correspondence is not real→real.

Everything depends therefore, on the existence of a point O which will project $A_1 \ldots A_5$ into points on the corresponding section of Ω by their opposite faces; in other words, O is an intersection of the 5 cones with vertices at $A_1 \ldots A_5$ respectively and standing on the sections of Ω by their opposite primes. Now the first of these *has* $A_1 \alpha_1$ *for a singular line and passes through* $A_2 \ldots A_6$ *and also* through $\alpha_2 \ldots . \alpha_6$. Thus A_1 and α_1 count each as 2 intersections of the 5 cones. For the second cone $A_2 \alpha_2$ is a double line and it is thus seen that out of the 32 intersections of the 5 cones in [5], $A_1 \ldots A_5$ $\alpha_1 \ldots \alpha_5$ count each for 2 intersections and A_6 and α_6 as single intersections, thus absorbing 22 intersections. There are therefore 10 more intersections, any one of which can be taken to be the point O. This would seem to establish the existence of Hart pentads of spheres in [3].

Similar considerations hold good in space of dimensions $n + 2 >$ or $= 5$. Hence the existence of proper † in-circum simplexes in a quadric in 3 or higher dimensions will lead to the existence of Hart systems of spheres.

We shall next take up for consideration the

11. Spheres in [n] which cut $n+2$ given spheres at the same angle.

We represent the given system of $n+2$ oriented spheres by points $A_1 \ldots A_{n+2}$ on the Lie quadric Ω in $[n+2]$.

These points lie on a prime whose pole with respect to Ω represents a turbine which is conjugate to all the given spheres. Hence

---

† *ie.* Simplexes for which the 2 sets A_r and α_r do not coalesce as happens for example in the case of a simplex referred to which Ω is $x_1 x_2 + x_3 x_4 + x_5 x_6 = 0$

its outer sphere cuts all the given spheres as the same angle while its inner sphere has with each of the given spheres the same tangential segment. If we reverse the orientation of one or more of the given spheres A_r we shall obtain other turbines conjugate to the spheres. Since reversing the orientation of all the spheres will give merely the turbine itself reversed, we see that if the given spheres are non-oriented, there are 2^{n+1} spheres each of which cuts them all at the same angle and also 2^{n+1} spheres concentric with the former which have each the same tangential segment with them (direct in some cases transverse in the others).

If we wish to study one of these sets, say the former, we may use cosine coördinates. Here the "coordinates" of a sphere are the cosines of the angles at which it cuts the given oriented spheres A_r. These cosine coordinates x_r satisfy a quadratic identity (vide vii p. 106).

$$x_1^2 \sec^2 \omega_1 + \ldots + 2\gamma_{12}\, x_1\, x_2 \sec \omega_1 \sec \omega_2 + \ldots = 1 \quad \ldots \quad (11{\cdot}1)$$

or briefly

$$f(x_1, \ldots x_{n+2}) \equiv \sum a_{rs}\, x_r\, x_s = 1 \quad \ldots \quad (11{\cdot}2)$$

where

$$a_{rr} = \sec^2 \omega_r,\ a_{rs} = \gamma_{rs} \sec \omega_r \sec \omega_s \quad \ldots \quad (11{\cdot}3)$$

and ω_r is the angle between A_r and the sphere A_r'' orthogonal to all the spheres except A_r and γ_{rs} is the cosine of the angle between A_r'' and A_s''. If the orientation of the sphere be reversed all the cosine coordinates change their sign, and the ratio of the coordinates remains unaltered. Hence cosine coordinates may be considered as the homogeneous coordinates of the non-oriented sphere in a representative $[n+1]$. Since for a point sphere the cosine coordinates are infinite, the Mobius quadric in this space has the equation $f(x_1 \ldots x_{n+2}) = 0$.

The equiangular spheres of the given $n+2$ spheres have cosine coordinates which are equal. Hence they may be taken as 1, $\pm$ 1, $\pm$ 1 etc. Hence there are 2^{n+1} spheres of this type which separate into two distinct systems (one having an odd number and the other an even number of negative signs) when n is even and belong to one system when n is odd. The two systems have the property that their respective points are in $(n+2)$-ple perspective with the vertices of the reference simplex as centres of perspective; or to put it geometrically, the coaxal system formed by A_r'' and any member of one system contains also a member of the other system. Even if n is odd the coaxal system formed by any A_r'' and any equiangular circle contains another member.

The angle α at which a particular member say $-1, 1, \ldots 1$ cuts the given spheres is given by (11·1) with $x_1 = -\cos\alpha$, $x_2 = \cos\alpha$ etc.

$$\therefore \qquad f(-1, 1, \ldots 1) = \sec^2\alpha \qquad \ldots \text{(11·4)}$$

Hence the 2^{n+1} equiangular spheres will cut at different angles α unless the given spheres are orthogonal so that f consists of square terms only, i.e. $a_{rs} = 0$ $r \neq s$. In this case all the 2^{n+1} spheres cut the given spheres at the same angle α given by

$$\sec^2\alpha = \sum \sec^2\omega_r = n+2.$$

When $n=2$, this gives 60° as the angle of intersection. In this case we have 3 tetrads of orthogonal circles of which each circle of one system cuts all the circles of the other system at 60°.*

Coming again to the general case, the points $1, \pm 1, \pm 1 \ldots$ representing the equiangular spheres of the set A may be specified as the common points of quadrics in $[n+1]$ self polar with respect to the polar set A'' (representing the orthogonal circles) and outpolar to the Absolute $f(x_1 \ldots x_{n+2}) = 0$ representing point spheres. Of these let us select $n+2$ spheres, namely

$$\begin{aligned} B_1 &= (-1, 1, 1, \ldots\ldots 1) \\ B_2 &= (1, -1, 1, \ldots\ldots 1) \text{ etc.} \\ B_{n+2} &= (1, 1, 1, \ldots\ldots -1) \end{aligned}$$

If every one of these should cut every sphere A at the same angle α, we should have

$$\sec^2\alpha = f(-1, 1, 1, \ldots 1) = f(1, -1, \ldots 1) = f(1, 1, \ldots -1) \text{ etc.}$$

Let us write ... (11·5)

$$s_1 = a_{12} + a_{13} + \cdots a_{1, n+2}$$
$$s_2 = a_{21} + a_{23} + \cdots a_{2, n+2} \text{ etc.}$$

where $a_{rs} = a_{sr}$ are the coefficients in (11·2). The equations (11·5) lead to

$$\sec^2\alpha = f - 4s_1 = f - 4s_2 = \cdots = f - 4s_{n+2} \qquad \ldots \text{(11·6)}$$

where $f = f(1, 1, \ldots 1)$. Hence it follows that

$$s_1 = s_2 = \cdots\cdots = s_{n+2} = \text{say } s$$

so that $f = a_{11} + a_{22} \cdots + a_{n+2, n+2} + (n+2)\, s$ which together with (11·6) leads to

---

* Vide Vaidyanathaswamy: On four mutually orthogonal circles, *Proc. of the Camb. Phil. Soc.* vol. 24 part 4.

$$\sec^2 \alpha = a_{11} + a_{22} + \cdots + a_{n+2,\, n+3} + (n-2)\, s$$

It is thus seen that 2-dimensional space is peculiar in that s removes itself, leading to

$$\sec^2 \alpha = \sec^2 \omega_1 + \sec^2 \omega_2 + \sec^2 \omega_3 + \sec^2 \omega_4 \qquad \ldots \quad (11{\cdot}7)$$

The conditions $s_1 = s_2 = s_3 = s_4$ are easily seen to be equivalent to $a_{12} = a_{34}$, $a_{13} = a_{24}$, $a_{14} = a_{23}$ or from (11·3),

$$\frac{\gamma_{12}}{\cos \omega_1 \cos \omega_2} = -\frac{\gamma_{34}}{\cos \omega_3 \cos \omega_4}\text{; and 2 similar relations}\dagger \qquad \ldots \quad (11{\cdot}8)$$

The results for 2 dimensions may therefore be summarised as follows:

> Let A_r be a set of 4 circles and let their 8 equiangular circles be denoted by B_r'' and C_r'' (these corresponding to sets with an odd and with an even number of negative signs). Let A'', B, C denote the tetrads orthogonal to A, B'' and C'', so that we have the scheme
>
> $$\begin{matrix} A & B & C \\ A'' & B'' & C'' \end{matrix}$$
>
> All quadrics through B'', C'' are outpolar to the absolute $f = 0$. If the circles B'' and A cut at equal angles, then all quadrics through A, C will also be outpolar to the absolute, and the condition for this to happen is that the set A should satisfy (11·8). The common angle of intersection α is then given by (11·7). The additional condition for a Hart-tetrad is given by putting $\alpha = 0$ in (11·7).

These results are scrappy and somewhat unrelated to the main body of known work in the field, which goes to show that the investigation has not penetrated deep enough yet.

---

† The tetrad A_r has given orientations and the orthogonal tetrad A_r'' may be oriented any way we please so as to definitize the various cosines which occur. If the orientation of any one—say A_1'' be reversed, γ_{1r} and $\cos \omega_r$ both change their signs leaving the equations (11·8) unaffected.

ON CORRELATION CONSTANTS IN MINGLED RECORDS

B. SITARAMAN, B.Sc. (Hon.)

Whenever a number of statistical records are mingled together, it is desirable to find out whether the single statistical constant derived from the mingled record represents or not the general tendency exhibited by similar constants, derived from the different records separately. S. Subramaniam † has discussed the case of the correlation coefficient and has shown that cases may arise where two characteristics are positively correlated in each record separately while the mingled record exhibits a negative correlation. The object of the present paper is to discuss the behaviour of the regression, coefficient as well as the intra-class correlation coefficient.

Let there be k records, and let x and y be the variables. Taking the sth record, let N_s, M_{sx}, M_{sy}, σ_{sx}, σ_{sy}, γ_s, b_s $[=\gamma_s\sigma_{sy}/\sigma_{sx}]$ be the number of observations, the means, the standard deviations, the correlation coefficient, and the regression coefficient respectively; and let the constants for the whole set after mingling be N, M_x, M_y, σ_x, σ_y, γ b $[=\gamma\sigma_y/\sigma_x]$ respectively. We then have the following relations:—

$$N=\sum_{s=1}^{k} N_s.$$

$$M_x=\frac{\sum N_s M_{sx}}{N};\quad M_y=\frac{\sum N_s M_{sy}}{N}$$

$$N\sigma_x^2=\sum N_s\sigma_{sx}^2+\sum N_s(M_{sx}-M_x)^2$$

$$N\sigma_y^2=\sum N_s\sigma_{sy}^2+\sum N_s(M_{sy}-M_y)^2$$

$$\gamma=\frac{\sum N_s\gamma_s\sigma_{sx}\sigma_{sy}+\sum N_s(M_{sx}-M_x)(M_{sy}-M_y)}{N\sigma_x\sigma_y}\qquad\ldots\quad(1)$$

Now $b=\gamma\,\sigma_y/\sigma_x$

$$=\frac{\sum N_s\gamma_s\sigma_{sx}\sigma_{sy}+\sum N_s(M_{sx}-M_x)(M_{sy}-M_y)}{\sum N_s\sigma_{sx}^2+\sum N_s(M_{sx}-M_x)^2}$$

$$=\frac{\sum N_s b_s\sigma_{sx}^2+\sum N_s(M_{sx}-M_x)(M_{sy}-M_y)}{\sum N_s\sigma_{sx}^2+\sum N_s(M_{sx}-M_x)^2}$$

---

† S. Subramaniam *The Mathematics Student* Vol. V., No. 3.

Let R, B and S_x be the statistical constants corresponding to the sample formed by the pairs of means $(M_{1x}, M_{1y}) \ldots\ldots (M_{kx}, M_{ky})$ with the frequencies $N_1, N_2, \ldots, N_k$ respectively. Then it follows that

$$b = \frac{\sum N_s b_s \sigma_{sx}^2 + NBS_x^2}{\sum N_s \sigma_{sx}^2 + NS_x^2} \qquad \ldots \quad (2)$$

Therefore, if we take k records each involving two variables x and y,

the regression coefficient of y upon x of the mingled record is the weighted arithmetic mean of the $(k+1)$ regression coefficients of the k records and the set of means, the weights being the corresponding sums of squares of x in the $(k+1)$ sets.

In the particular case when $b_1 = b_2 = \cdots b_k = \lambda$, b is also equal to λ, only when

$$\lambda = \frac{\sum N_s \lambda\sigma_{sx}^2 + NBS_x^2}{\sum N_s \sigma_{sx}^2 + NS_x^2}$$
$$= \frac{\lambda \sum N_s \sigma_{sx}^2 + NBS_x^2}{\sum N_s \sigma_{sx}^2 + NS_x^2}$$

i.e. $\lambda \sum N_s \sigma_{sx}^2 + \lambda NS_x^2 = \lambda \sum N_s \sigma_{sx}^2 + NBS_x^2$

i.e. $N(\lambda - B) S_x^2 = 0 \qquad \ldots \quad (3)$

$\therefore$ Either $\lambda = B$ or $S^2_x = 0$, since $N \neq 0$.

Hence,

if the regression coefficient of y upon x is the same for each of k records separately, it is unaffected by mingling if either

(1) *it is equal to the regression coefficient of the sample formed by the means of the k records, or*

(2) *the means of x in the several records are all equal.*

We shall now consider the case of the intra-class correlation *i.e.* correlation between two variables of identical character. Here, if (x_{st}, y_{st}) is a pair of values, (y_{st}, x_{st}) is another pair, so that the mean and the standard deviation are the same for each variable; and the intra-class correlation coefficient is given by

$$\gamma_s = \frac{\sum_{t=1}^{N_s} (x_{st} - M_s)(y_{st} - M_s)}{N_s \sigma_s^2}$$

Now, the correlation coefficient after mingling is given by

$$\gamma = \frac{\sum N_s \gamma_s \sigma_{sx} \sigma_{sy} + \sum N_s (M_{sx} - M_x)(M_{sy} - M_y)}{N\sigma_x \sigma_y}$$

But here $M_{sx} = M_{sy}$ ($= M_s$) and $\sigma_{sx} = \sigma_{sy} = \sigma_s$; so we have on substitution

$$\gamma = \frac{\sum N_s \gamma_s \sigma_s^2 + NS^2}{\sum N_s \sigma_s^2 + NS^2} \qquad \ldots \quad (4)$$

Since the correlation coefficient is always less than or equal to unity,

$$|\sum N_s \gamma_s \sigma_s^2| \leqslant \sum N_s \sigma_s^2 \qquad \ldots \quad (5)$$

Hence, by the relation $\frac{p}{q} \leqslant \frac{p+\mu}{q+\mu}$ where μ and q are positive and $|p| \leqslant q$, we have from relations (4) and (5)

$$\frac{\sum N_s \gamma_s \sigma_s^2}{\sum N_s \sigma_s^2} \leqslant \gamma \leqslant 1 \qquad \ldots \quad (6)$$

i.e. the intra-class correlation coefficient after mingling lies between unity and the weighted mean of the coefficients of the several records, the weights being the corresponding 'sums of squares'. It attains the value unity when the correlation coefficient of each record is also unity.

Further, since $NS^2 \geqslant 0$, the sign of γ depends upon $\sum N_s \gamma \sigma_s^2$. *Therefore if positive intra-class correlation is exhibited in each of the k records, then the mingled record must also exhibit a positive intra-class correlation.* On the other hand, if intra-class correlation is negative in each record, the mingled record will also exhibit a negative correlation only when $NS^2 < |\sum N_s \gamma_s \sigma_s^2|$. Otherwise we have the case of a mingled record showing a positive intra-class correlation, when each record exhibits a negative intra-class correlation separately.

AN APPROXIMATION CONNECTED WITH EXP. *x*

DR. S. CHOWLA AND F. C. AULUCK, *Lahore.*

1. Ramanujan proved that $S(x) \sim \frac{1}{2} e^x$ where

$$S(r) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^r}{r!};$$

but the proof as given by him in the *Collected Papers* (p. 324) is not rigorous. In this note we give a rigorous proof of the above theorem. It may be remarked that the whole contribution of $S(x)$ is from a *small* interval of r in the neighbourhood* of x. In fact

$$S(mx) = o\ (e^x) \qquad \text{if } m < 1 \text{ and } S(mx) \sim e^x \text{ if } m > 1.$$

---

* Hardy and Wright. *An Introduction to the Theory of Numbers* (1938) p. 360.

Polya and Szego, *Aufgaben und Lehrsätze aus der Analysis*, Erster Band (1925) p. 80.

Ramanujan's problem was to prove that y lies between $\frac{1}{3}$ and $\frac{1}{2}$ where y satisfies the equation

$$\frac{1}{2}e^{x}=1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^x}{x!}y.$$

A 'partial proof' was given by him in which he proved that $y=1/2$, when $x=0$ and $y\rightarrow 1/3$ when $x\rightarrow\infty$. Later proofs were supplied by Watson(1) and Szego(2). Watson gave some reasons for believing that

$$y=\frac{1}{3}+\frac{1}{135\,(x+k)}$$

where k lies between $\frac{8}{45}$ and $\frac{2}{21}$, but no complete proof has been given for the above statement.

2.
$$S(x)=\frac{x^x}{x!}\int_0^\infty e^{-z}\left(1+\frac{z}{x}\right)^x dz$$

$$=\frac{x^x}{x!}\left\{\int_0^{x\phi}+\int_{x^\phi}^{2x/3}+\int_{2x/3}^{\infty}\right\}$$

$$=\frac{x^x}{x!}\left\{I_1+I_2+I_3\right\}$$

where $\frac{1}{2}<\varphi<\frac{2}{3}$.

In I_3, $z>\frac{2x}{3}$, $1+\frac{z}{x}\leqslant\frac{5}{2}\frac{z}{x}$; hence

$$0<I_3<\int_0^\infty e^{-z}\left(\frac{5}{2}\right)^x\left(\frac{z}{x}\right)^x dz=\frac{x!}{x^x}\left(\frac{5}{2}\right)^x$$

In I_2, the integrand is

$$\exp\left\{-z+x\log\left(1+\frac{z}{x}\right)\right\}$$

$$=\exp\left\{-\frac{z^2}{2x}\left(1+\frac{\theta z}{x}\right)^{-2}\right\}\ (0<\theta<1)$$

$$\leqslant\exp\left\{-\frac{1}{2}x^{2\phi-1}\left(1+\frac{2}{3}\right)^{-2}\right\}$$

$$=\exp\left(-\frac{9}{50}x^{2\phi-1}\right)$$

Hence $I_2\rightarrow 0$ as $x\rightarrow\infty$ since $\phi>\frac{1}{2}$

---

(1) *Proc. of London Math. Soc.* Ser. 2. Vol 29. Part 4 (1928) 293—308.

(2) *Journal of the London Math. Soc.* 3 (1928) 225—232.

Finally $I_1 = \int_0^{x^{\varphi}} \exp\left\{ -\frac{z^2}{2x} + \frac{\theta z^3}{x^2} \right\} dz.$ $(0 < \theta < 1)$

$$= \sqrt{2x} \int_0^{G} \exp\left\{ -u^2 + \frac{\theta 2\sqrt{2}\, u^3}{\sqrt{x}} \right\} du$$

where $z = (\sqrt{2}\,\sqrt{x}\,u)$ and $G = x^{\phi - \frac{1}{2}}\,\sqrt{2}$.

$$\text{But } \exp\left\{ \frac{2\sqrt{2}\,\theta u^3}{\sqrt{x}} \right\} - 1 + O\left(\frac{u^3}{\sqrt{x}} \right)$$

$$= 1 + O\left(x^{3\phi - 2} \right)$$

Since $3\varphi < 2$, it follows that

$$I_1 \sqrt{2x} \to \int_0^{\infty} e^{-u^2}\, du = \frac{\sqrt{\pi}}{2}$$

Hence $S(x)\, e^{-x} \backsim \frac{x^x}{x!}\, e^{-x}\, \frac{\sqrt{\pi}}{2}\, \sqrt{2x} \backsim \frac{1}{2}$,

since by Stirling's Theorem $x! \backsim \sqrt{2\pi x}\,(x/e)^x$.

3. $\frac{x^r}{r!}$ is maximum at $r = x$. Hence, if $0 < m < 1$.

$$S(mx) < mx\, \frac{x^{mx}}{\Gamma(mx)}$$

$$\backsim \frac{mx\; x^{mx}}{\sqrt{(2\pi m x)}}\, (mx/e)^{-mx}$$

$$\backsim \sqrt{\frac{mx}{2\pi}}\, (e/m)^{mx}$$

As $(e/m)^m < e$ if $0 < m < 1$,

$S(mx)\, e^{-x} \to 0$

Finally we notice that

$$\sum_{r=x+1}^{\infty} \frac{x^r}{r!} \backsim \frac{1}{2} e^x$$

$$\text{or } \frac{x}{x+1} + \frac{x^2}{(x+1)(x+2)} + \cdots \backsim \sqrt{(\tfrac{1}{2}\pi x)}$$

If $m > 1$ and assuming mx to be integral

$$\frac{x^{x(m-1)+1}}{(x+1)(x+2)\cdots(mx+1)} + \frac{x^{x(m-1)+2}}{(x+1)(x+2)\cdots(mx+2)} + \cdots$$

$$= \frac{x^{x(m-1)+1}}{(x+1)(x+2)\cdots(mx+1)} \left\{ 1 + \frac{x}{mx+2} + \frac{x^2}{(mx+2)(mx+3)} + \cdots \right\}$$

$$< 1 + \frac{1}{m} + \frac{1}{m^2} + \cdots = \frac{m}{m-1}.$$

Hence $S(mx)\, e^{-x} - 1 \to 0$.

ON WEIRESTRASS APPROXIMATION THEOREM

DR. S. CHOWLA AND F. C. AULUCK

1. It is known* that any function $f(x)$, continuous in an interval $a \leqslant x \leqslant b$ can be approximated uniformly through a polynomial $P(x)$; that is, the difference $|f(x) - P(x)|$ can be made less than ε which is independent of x. In the present note we prove that

$$n = O\left(\varepsilon^{-2/k} \log 1\ \varepsilon\right)$$

where n is the degree of $P(x)$ and $f(x)$ satisfies Lipschitz condition $f(x+\theta) - f(x) = O(\theta^k)$, k being a positive number, independent of x. It follows that for all differentiable functions

$$n = O\left(\varepsilon^{-2} \log 1\ \varepsilon\right)$$

It may be mentioned that most of the ordinary functions like $\sin x$, $\cos x$, e^x, require only $O\left(\dfrac{\log 1\ \varepsilon}{\log\log 1/\varepsilon}\right)$ terms.

2. For the sake of simplicity, we take the interval (a, b) to lie wholly within the interval $(0,1)$. Consider the integrals

$$J_n = \int_0^1 (1-v^2)^n\, dv, \quad J_n^* = \int_\delta^1 (1-v^2)^n\, dv$$

δ being a fixed number in the interval $(0,1)$.

Since $\qquad J_n > \int_0^1 (1-v)^n\, dv = \dfrac{1}{n+1}$

and $\qquad J_n^* < (1-\delta^2)^n (1-\delta) < (1-\delta^2)^n$

therefore $\quad J_n^*/J_n < (n+1)(1-\delta^2)^n \qquad \ldots \quad (1)$

$$P_n(x) = \frac{\int_\alpha^\beta f(u)\,[1-(u-x)^2]^n\, du}{\int_{-1}^1 (1-u^2)^n\, du} \quad (0 < \alpha < a < b < \beta\ 1) \qquad \ldots \quad (2)$$

is a polynomial in x of degree $2n$.

$$\text{The numerator} = \int_{\alpha-x}^{\beta-x} f(v+x)(1-v^2)^n\, dv$$

$$= \int_{\alpha-x}^{-\delta} + \int_{-\delta}^{\delta} + \int_{\delta}^{\beta-x}$$

$$= I_1 + I_2 + I_3$$

---

* Courant and Hilbert, *Methoden der Mathematischen Physik I.* (1931) pp. 55—57.

If M is the maximum of $|f(x)|$ in $\alpha \leqslant x \leqslant \beta$

$$|I_1| < M \int_{-1}^{-\delta} (1-v^2)^n \, dv = M \; J_n^*$$

$$|I_3| < M \int_{\delta}^{1} (1-v^2)^n \, dv = M \; J_n^*$$

$$I_2 = f(x)\int_{-\delta}^{\delta} (1-v^2)^n \, dv + \int_{-\delta}^{\delta} [f(v+x)-f(x)] \, (1-v^2)^n \, dv$$

$$= 2\, f(x)(J_n - J^*_n) + \int_{-\delta}^{\delta} [f(v+x)-f(x)] \, (1-v^2)^n \, dv$$

Assume that $|f(v+x)-f(x)| < Av^k$ where A and k are positive and independent of v. So

$$|f(v+x)-f(x)| < \varepsilon/2$$

if $a < x < b$, $|v| < \delta < \left(\frac{\varepsilon}{2A}\right)^{1/k}$ ... (3)

Therefore $\left| \int_{-\delta}^{\delta} [f(v+x)-f(x)] \, (1-v^2)^n \, dv \right|$

$$< (\varepsilon/2)\int_{-\delta}^{1} (1-v^2)^n \, dv$$

$$< (\varepsilon/2)\int_{-1}^{1} (1-v^2)^n \, dv$$

$$= \varepsilon \; J_n$$

Hence $|f(x) - P_n(x)| < 2M \; J_n^*/J_n + \varepsilon/2$

$$< 2(n+1)(1-\delta^2)^n \; M + \varepsilon/2$$

$$< \varepsilon$$

if $(n+1)(1-\delta^2)^n < \varepsilon/4M$ ... (4)

or $n \log \frac{1}{1-\delta^2} - \log (n+1) > \log 4M/\varepsilon.$

and therefore $n = O\left(\frac{1}{\delta^2} \log 1/\varepsilon\right)$

$$= O\,(\varepsilon^{-2/k} \log 1/\varepsilon) \text{ by (3)}$$

3. In general the coefficients of the polynomial $P(x)$ are functions of n. For example consider the function $1/x$ $(1 \leqslant x \leqslant 2)$

Substituting $x = \frac{3}{2} + y$ $(-\frac{1}{2} \leqslant y \leqslant \frac{1}{2})$

we obtain $\frac{1}{x} = \frac{1}{\frac{3}{2}+y}$

$$= \frac{2}{3}\left\{1 - \frac{2y}{3} + \left(\frac{2y}{3}\right)^2 \cdot \cdot \cdot \cdot + (-1)^{n-1}\left(\frac{2y}{3}\right)^{n-1}\right\} + k_n$$

$$= P_n(x) + k_n$$

where $|k_n| = \frac{2}{3} \cdot \left(\frac{2y}{3}\right)^n / \left(1 + \frac{2y}{3}\right)$

$$< \varepsilon \text{ if } n = O\,(\log 1/\varepsilon)$$

Obviously the coefficients of $P_n(x)$ are functions of n.

NOTES AND DISCUSSIONS

Heliacal rising of the Moon

The moon's visibility for the first time in the evening after new moon determines the beginning of the Hijri month. The problem is of some interest to this Observatory as the day of the heliacal rising of the moon fixes the commencement of the several months in the Hijri calendar. As this phenomenon is to a great extent dependent on the times of sunset and moonset, the duration of twilight, the state of the sky etc. it is difficult to lay down hard and fast rules for determining the day on which the moon can be seen in critical cases.

I may note in passing, that the problem will be of special interest, from the religious point of view, to persons belonging to the *Bodhayana Sutra* as the moon must be visible on the third day of their observing the *Amavasya.* According to the Hindus, if the first *Tithi* ends before the 25th *ghati* or in other words if the elongation of the moon be 12 degrees two hours before sunset, the appearance of the thin crescent is certain. If the *tithi* ends between the 25th and 35th *ghatis*, the appearance is doubtful and after the 35th *ghati* it is impossible.

According to another and a better method, the method of *Kalamsas* as it is called, the interval between the times of sunset and moonset is calculated; and if it exceeds 50 minutes, visibility is believed to be certain.

According to the late Mr. Ketkar, the visibility of the moon is certain if her altitude is greater than 12 degrees. He gives the following rule for determining the moon's visibility. If z denotes the zenith distance of the nonagesimal, e the elongation of the moon at sunset and l the latitude of the moon, visibility of the moon will be certain, doubtful or impossible according as $e \cos z + l \sin z > =$ or <12 degrees respectively. (z is treated as positive when south of the zenith).

The limit of 12 degrees is rather too much. Further the above expression for altitude is only a very rough approximation. The more rigorous expression for the altitude based on the zenith distance of the nonagesimal is derived as follows :—

Let Z denote the zenith

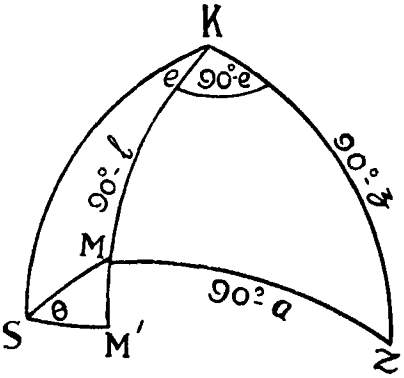
K the pole of the ecliptic

M the moon
S the sun at sunset
M′ the foot of the secondary to the ecliptic through M
a the altitude of the moon
l the latitude of the moon
z the zenith distance of the nonagesimal
SKM $= e$, the elongation of the moon
$MSM' = \theta$
Then $SKZ = 90^\circ$
$\therefore \quad MKZ = 90^\circ - e$
From the Δ MKZ we have,

$$\sin a = \sin l \sin z + \cos l \cos z \sin e$$
$$= \sin SM \sin \theta \sin z + \sin SM \cos \theta \cos z$$
$$= \sin SM \cos (z - \theta)$$
$$a = \mathrm{Sin}^{-1} [\sin SM \cos (z - \theta)].$$

where $SM = \cos^{-1} (\cos l \cos e)$

and $\theta = \tan^{-1} [\tan l \operatorname{cosec} e]$



If ϕ denotes the latitude of the place and ω the obliquity of the ecliptic, z lies between $\phi + \omega$ and $\phi - \omega$; also the maximum numerical value of θ is 23° 25′ taking 5° 9′ for the mean value of l and 12° for e.

a is minimum when SM is minimum and $z - \theta$ maximum (i.e.) z maximum and θ minimum. In Hyderabad, for example, the maximum value of z is 40 53′ so that the minimum value of a can be as low as 5 or 6° though the moon's elongation may be 12° or 13°. This will occur in the most unfavourable case when the ecliptic is least inclined to the horizon, as at the time of the autumnal equinox and the moon has the maximum southern latitude. Though this may be of rare occurrence, it shows that the elongation criterion is not reliable for determining the visibility of the moon.

After examining a number of critical cases I have found that the altitude criterion is the safest for determining the visibility of the moon. If her altitude be greater than 11° we can safely predict the appearance of the crescent in our latitudes; when the altitude lies between 10° and 11° the appearance is doubtful and has to be decided by considering other factors such as the duration of twilight etc. When it is less than 10°, the appearance is impossible.

Nizamiah observatory,
Hyderabad—Deccan.

M. V. VAIDYANATHA SASTRY.

A Note on Cauchy's Theorem

Cauchy's Theorem:—If $f(z)$ is holomorphic inside and continuous on the closed contour C, then $\int_C f(z)\,dz = 0$. From the given conditions we get,

(i) To a given ε corresponds an η such that $\left| \frac{f(z) - f(z')}{z - z'} - f'(z) \right| < \varepsilon$ if $|z - z'| < \eta$, η being independant of the positions of the two points z and z' inside or on a closed contour γ lying inside C. (This is called Goursat's Lemma).

(ii) To a given ε' corresponds an η' such that $| f(z) - f(z)| < \varepsilon'$ if $|z - z'| < \eta'$, η' being independant of the positions z and z' inside or on C. (Uniform continuity).

(iii) $\int_\gamma f(z)\,dz + \int_\gamma zf'(z)\,dz = 0$, γ being entirely inside C.

To prove this last result, let δ be the greatest distance of the origin from C, and l be the length of the contour γ. Then

$$\int_\gamma f(z)\,dz = \lim \sum (z_i - z_{i-1})\, f(z_i)$$

$$= -\lim \sum z_i\, [f(z_{i+1}) - f(z_i)] = -\lim \sum z_i\, [f'(z_i) + \varepsilon_i](z_{i+1} - z_i)$$

$$\therefore\ \left| \int_\gamma f(z)\,dz + \int_\gamma zf'(z)\,dz \right| = \left| -\sum z_i\, \varepsilon_i\, (z_{i+1} - z_i) \right| < \delta\varepsilon l$$

since $|\varepsilon_i| < \varepsilon$ ultimately by (i)

Proof:—We take the origin to be any convenient point inside C. We assume that the curve is such that any vector from O, the origin meets the curve in only one point.

If P be any point on C divide OP into n equal parts at $P_1, P_2, \ldots P_{n-1}$. Then as P moves on C, P_r describes a similar curve C_r, $r = 1, 2 \ldots, n-1$. Take n greater than δ/η and $\delta\,\eta'$. Let L be the length of the curve C and M be the upper bound of $f(z)$ inside and on C.

If P is z_i then P_r is $z_i\ r\ n$

Also $f\left(z_i \frac{r+1}{n}\right) - f\left(z_i \frac{r}{n}\right) = \frac{z_i}{n}\left[f'\left(z_i \frac{r}{n}\right) + \varepsilon_i\right]$, $|\varepsilon_i| < \varepsilon$ by (i)

$$\therefore\ \int_{C_{r+1}} f(z)\,dz = \lim \sum \frac{r+1}{n}\left(z_i - z_{i-1}\right) f\left(z_i \frac{r+1}{n}\right)$$

$$= \lim \sum \frac{r+1}{n}\left(z_i - z_{i-1}\right)\left[f\left(z_i \frac{r}{n}\right) + \frac{z_i}{n} f'\left(z_i \frac{r}{n}\right) + \varepsilon_i \frac{z_i}{n}\right]$$

$$= \frac{r+1}{r}\int_{C_r} f(z)\,dz + \frac{r+1}{r^2}\int_{C_r} zf'\,dz + \sum \frac{r+1}{n}\,\varepsilon_i\, \frac{z_i}{n}\left(z_i - z_{i-1}\right)$$

$$\therefore \quad \int_{C_{r+1}} f dz - \int_{C_r} f dz = -\frac{1}{r^2}\int_{C_r} f dz + \frac{r+1}{n^2} \sum z_i\, \varepsilon_i\, (z_i - z_{i-1}) \text{ by (iii)}$$

$$\therefore \quad \left| \int_{C_{r+1}} - \int_{C_r} \right| \leqslant \frac{1}{r^2} \left| \int_{C_r} f dz \right| + \frac{r+1}{n^2} \left| \sum z_i\, \varepsilon_i\, (z_i - z_{i-1}) \right|$$

$$< \frac{1}{r^2}\, \mathrm{M} \left(\frac{\mathrm{L}r}{n}\right) + \frac{r+1}{n^2}\, \varepsilon\delta\, \mathrm{L} \leqslant \frac{\mathrm{ML}}{rn} + \frac{\varepsilon\delta \mathrm{L}}{n}$$

$$\left| \int_{C_1} f dz \right| \leqslant \frac{\mathrm{ML}}{n}$$

$$\left| \int_{C} - \int_{C_{n-1}} \right| = \left| \sum (z_i - z_{i-1}) f(z_i) - \sum \frac{n-1}{n} (z_i - z_{i-1}) f\left(z_i \frac{n-1}{n}\right) \right|$$

$$\leqslant \left| \sum (z_i - z_{i-1}) \left[f(z_i) - f\left(z_i \frac{n-1}{n}\right) \right| + \frac{1}{n} \left| \sum (z_i - z_{i-1}) f\left(z_i \frac{n-1}{n}\right) \right.$$

$$\leqslant \mathrm{L}\, \varepsilon' + \frac{\mathrm{ML}}{n}$$

$$\therefore \quad \left| \int_{C} f dz \right| < \left| \int_{C} - \int_{C_{n-1}} \right| + \sum_{r=1}^{n-2} \left| \int_{C_{r+1}} - \int_{C_r} \right| + \left| \int_{C_1} \right|$$

$$< \mathrm{L}\, \varepsilon' + \frac{\mathrm{ML}}{n} + \frac{\mathrm{ML}}{n} \left(\sum_{r=1}^{n-2} \frac{1}{r} \right) + \varepsilon\delta\, \mathrm{L} + \frac{\mathrm{ML}}{n}$$

Making $n \to \infty$ the left hand side is arbitrarily small. Now any region enclosed by a simple curve can be divided into a finite number of regions in each of which we can find a point such that any vector from the point meets its contour once only, and sum of the integrals round such contours vanishes.

St. Joseph's Petit Seminary,
Ernakulam.

A. KOMULLY.
(K. O. AUGUSTINE).

Lecture Notes

I—Reduction of Cylinders and Paraboloids

§1. The general equation of a cylinder (other than parabolic) or a paraboloid can be put in the form

$$(lx + my + nz)(l'x + m'y + n'z) + 2ux + 2vy + 2wz + d = 0 \qquad \ldots \quad (1)$$

which may be re-written as

$$(lx + my + nz + \lambda)(l'x + m'y + n'z + \mu) = u'x + v'y + w'z + d'$$

where
$$u' = \lambda l' + \mu l - 2u$$
$$v' = \lambda m' + \mu m - 2v$$
$$w' = \lambda n' + \mu n - 2w$$
$$d' = \lambda\mu - d, \text{ and } (l', m', n') \neq (l, m, n) \qquad \ldots \quad (2)$$

(i) If $\begin{vmatrix} u & v & w \\ l & m & n \\ l' & m' & n' \end{vmatrix} = 0,$

find λ, μ so that $u' = v' = 0$ and therefore $w' = 0$.

Taking as two co-ordinate planes of reference ($\mathrm{X} = 0$, $\mathrm{Y} = 0$) the planes bisecting the angles between

$$lx + my + nz + \lambda = 0$$
$$l'x + m'y + n'z + \mu = 0$$

and for the third plane any plane perpendicular to these, the equation (1) reduces to the Standard form

$$\mathrm{X}^2 \sum (l + kl')^2 - \mathrm{Y}^2 \sum (l - kl')^2 = 4k\ (\lambda\mu - d) \qquad \ldots \quad (3)$$

where $k^2 = \sum l^2 / \sum l'^2$, and the equation generally represents a cylinder.

(ii) If $\begin{vmatrix} u & v & w \\ l & m & n \\ l' & m' & n' \end{vmatrix} \neq 0$, find λ, μ so that

$$u'l + v'm + w'n = 0 = u'l' + v'm' + w'n'$$

Then the plane $u'x + v'y + w'z + d' = 0$ will be perpendicular to $lx + my + nz + \lambda = 0$, and $l'x + m'y + n'z + \mu = 0$ and therefore to the planes bisecting the angles between these. With the bisecting planes as two co-ordinate planes and $u'x + v'y + w'z + d' = 0$ as the third co-ordinate plane, our original equation will reduce to the standard form

$$\mathrm{X}^2 \sum (l + kl')^2 - \mathrm{Y}^2 \sum (l - kl')^2 = 4k\mathrm{Z}\ (\sum u'^2)^{\frac{1}{2}} \qquad \ldots \quad (4)$$

k having the same significance as in (3).

The above method is sufficient for numerical work, provided the terms of the second degree in (1) factorise rationally.

§2. If the equation of the conicoid be given by

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$$

we may express the coefficients in the reduced equations (3) and (4) in terms of those of the given equation.

In the case of the cylinder, the calculation may be made thus:

$$\sum (l \pm kl')^2 = 2k \left(\pm \sum ll' + \frac{\sum l^2}{k} \right) = 2k\, (\pm ll' + \mathrm{T})\ (\text{say}) \qquad \ldots \quad (5)$$

If $f(x,\ y,\ z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$

$$\equiv (lx + my + nz)(l'x + m'y + n'z)$$

we have

$$\left.\begin{aligned} a = ll', \quad b = mm', \quad c = nn' \\ 2f = mn' + m'n,\ 2g = nl' + n'l,\ 2h = lm' + l'm, \end{aligned}\right\} \qquad \ldots \quad (6)$$

$$\begin{aligned} T^2 &= \sum l^2 \sum l'^2 \\ &= (\sum ll')^2 + \sum (mn' - m'n)^2 \\ &= (\sum ll')^2 + \sum (mn' + m'n)^2 - 4 \sum mn\ m'n' \\ &= (\sum a)^2 + 4 \sum f^2 - 4 \sum ab \quad \text{by (6)} \end{aligned}$$

$$i.e.\quad T = (a^2 + b^2 + c^2 - 2ab - 2bc - 2ca + 4f^2 + 4g^2 + 4h^2)^{\frac{1}{2}} \qquad \ldots \quad (7)$$

λ, μ being given by $u' = v' = w' = 0$,

we have $\lambda\mu = \dfrac{4\,(um - vl)\,(um' - vl')}{-(lm' - l'm)^2}$ = two similar expressions

$$= \frac{4 \sum (bu^2 + av^2 - 2huv)}{-\sum (lm' - l'm)^2}$$

$$= \frac{\sum a \sum u^2 - f(u,\ v,\ w)}{\sum ab - \sum f^2} \qquad \ldots \quad (8)$$

The reduced equation is

$$(a + b + c + T)\,x^2 + (a + b + c - T)\,y^2 - 2\,(\lambda\mu - d). \qquad \ldots \quad (9)$$

Illustration :

$$2y^2 - 2yz + 2zx - 2xy - x - 2y + 3z - 2 = 0$$

$$i.e.\ (2y - 2z)(y - x) - x - 2y + 3z - 2 = 0$$

Here $\begin{vmatrix} -1 & -2 & 3 \\ 0 & 2 & -2 \\ -1 & 1 & 0 \end{vmatrix} = 0.$ Therefore the equation represents a cylinder.

$k^2 = \frac{8}{2} = 4$; we may take $k = 2$.

λ, μ are given by

$$-\lambda + \mu\cdot 0 = -1$$

$$\lambda + 2\mu = -2$$

Hence $\lambda = 1,\ \mu = -\frac{3}{2}$

The reduced equation is, therefore, given by

$$24X^2 - 8Y^2 = 8 \times \tfrac{1}{2} = 4$$

$$i.e.\quad 6X^2 - 2Y^2 = 1$$

Or, from (9) directly, since $a + b + c = 2$, $T = 4$, $\lambda\mu - d = 1/2$ we get the same equation.

In the case of paraboloids, (5), (6), and (7) hold good as before and we have to calculate only $\sum u'^2$.

Since $u'l+v'm+w'n=0,\ u'l'+v'm'+w'n'=0,$

we have

$$\frac{u'}{mn'-m'n}=\frac{v'}{nl'-n'l}=\frac{w'}{lm'-l'm}=\frac{(u'^2+v'^2+w'^2)^{\frac{1}{2}}}{\{\sum(mn'-m'n)^2\}^{\frac{1}{2}}}$$

$$=\frac{\sum u'(mn'-m'n)}{\sum(mn'-m'n)^2}$$

$$\therefore\ (\sum u'^2)^{\frac{1}{2}}=\{\sum(mn'-m'n)^2\}^{-\frac{1}{2}}\sum u'(mn'-m'n)$$

But from (2), $\sum u'(mn'-m'n)=-2\sum u(mn'-m'n)\neq 0$, by our original assumption.

Hence $\sum u'^2$ does not vanish and the equation (4) always represents a paraboloid.

From (6), $(mn'-m'n)$, $(nl'-n'l)$, $(lm'-l'm)$ are respectively proportional to A, H, G which are the cofactors of a, h, g respectively in the vanishing determinant $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$.

Hence $$(\sum u'^2)^{\frac{1}{2}}=-\frac{2(uA+vH+wG)}{(A^2+H^2+G^2)^{\frac{1}{2}}} \qquad \ldots (10)$$

The reduced equation of a paraboloid takes the form

$$(a+b+c+T)X^2+(a+b+c-T)Y^2=-\frac{4k(uA+vH+wG)}{(A^2+H^2+G^2)^{\frac{1}{2}}} \qquad \ldots (11)$$

Cor: $uA+vH+wG\neq 0$ for a paraboloid.

Illustration:

$$4y^2+4z^2+4yz-2x-14y-22z+33=0$$

Here the terms of the second degree do not factorise rationally and therefore the method of § 1 is not applicable easily.

We fall back on (11), using the values of T and A, H, and G.

We find $A=12$, $H=0$, $G=0$, $u=-1$ and $uA+vH+wG=-12$, $T=4$.

$\therefore$ The equation represents a paraboloid and the reduced form is $12x^2+4y^2=4z$ *i. e.* $3x^2+y^2=z$.

II Euler's Function ϕ (N)

If $p_1, p_2, \ldots p_n$ be the distinct prime factors of N, we shall prove the well-known relation

$$\phi(N) = N\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_n}\right)$$

by using two simple lemmas:

$$\text{(i)}\quad \phi(N) = p_1\,\phi\left(\frac{N}{p_1}\right) \text{ if } \frac{N}{p_1} \text{ contains } p_1 \text{ as a factor}$$

and $$\text{(ii)}\quad \phi(N) = (p_1-1)\,\phi\left(\frac{N}{p_1}\right) \text{ if } \frac{N}{p_1} \text{ is prime to } p_1,$$

where $\phi(m)$ denotes the number of positive integers less than m which are relatively prime to m.

The positive integers less than N are given without repetition by $a \cdot \frac{N}{p_1} + r$ for $a = 0, 1, \ldots p_1 - 1$ and $r = 0, 1, \ldots\ldots \left(\frac{N}{p_1} - 1\right)$, $a = r = o$ being excepted.

Evidently $a\frac{N}{p_1} + r$ is prime to $\frac{N}{p_1}$ and therefore to N in case (i), if and only if r is prime to N/p_1. Since every such value of r can be associated with the p_1 - values of a, there are $p_1\,\phi\,(N\,p_1)$ less than N and prime to N.

Hence, the first lemma is proved.

In case (ii), if r_1 be any integer prime to N/p_1 and less than N/p_1 the set of integers

$$r_1,\ \frac{N}{p_1} + r_1,\ 2\frac{N}{p_1} + r_1, \cdots (p_1-1)\frac{N}{p_1} + r_1$$

is congruent to the set of integers $0, 1, \cdots (p_1 - 1)$ rearranged, (mod p_1) and includes exactly $(p_1 - 1)$ integers not divisible by p_1 and therefore prime to p_1.

Thus, in this case, we can associate only $(p_1 - 1)$ values of a with each r_1 belonging to ϕ (N/p_1), so as to obtain integers prime to N/p_1 and p_1 and therefore prime to N.

Hence $\phi(N) = (p_1 - 1)\,\phi\left(\frac{N}{p_1}\right)$.

If $N = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$, we get by repeated applications of lemma (i),

$$\phi(N) = p_1 \phi(p_1^{\alpha_1-1} p_2^{\alpha_2} \cdots p_n^{\alpha_n})$$
$$= p_1^2 \phi(p_1^{\alpha_1-2} p_2^{\alpha_2} \cdots p_n^{\alpha_n})$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$= p^{\alpha_1-1} \phi(p_1, p_2^{\alpha_2} \cdots p_n^{\alpha_n})$$

and by lemma (ii),

$$\phi(p_1 p_2^{\alpha_2} \cdots p_n^{\alpha_n}) = (p_1 - 1)\,\phi(p_2^{\alpha_2} \cdots p_n^{\alpha_n})$$

Thus $\phi(N) = p_1^{\alpha_1-1}(p_1 - 1)\,\phi(p_2^{\alpha_2} \cdots p_n^{\alpha_n})$
$$= p_1^{\alpha_1-1}(p_1 - 1)\,\phi\left(\frac{N}{p_1^{\alpha_1}}\right)$$

Repeating the same process with $\phi\left(\frac{N}{p_1^{\alpha_1}}\right)$, $\phi\left(\frac{N}{p_1^{\alpha_1} p_2^{\alpha_2}}\right) \cdots\cdot$ and so on, we get

$$\phi(N) = p_1^{\alpha_1-1} p_2^{\alpha_2-1} \cdots p_n^{\alpha_n-1}(p_1 - 1)(p_2 - 1) \cdots (p_n - 1)$$
$$= N(1 - p_1^{-1})(1 - p_2^{-1}) \cdots (1 - p_n^{-1}).$$

Thus the theorem is proved.

III—Homogeneous Products

We give below a short proof of the result ${}_nH_r = {}^{n+r-1}C_r$. As in Chrystals' Algebra, Part II, p 11, we add to each of the r combinations all the n-letters $a_1\ a_2\ a_3 \cdots a_n$ and get products of the form $a_1^{\alpha_1}\ a_2^{\alpha_2} \cdots\cdots a_n^{\alpha_n}$, where $\sum_{i=1}^{i=n} \alpha_i = n + r$

Consider the ordered set of $(n-1)$ numbers

$$(\alpha_1, \alpha_1+\alpha_2, \alpha_1+\alpha_2+\alpha_3, \cdots\cdots \alpha_1+\alpha_2+\cdots+\alpha_{n-1}) \quad \ldots \quad (A)$$

which has one-to-one correspondence with the ordered set

$$(\alpha_1, \alpha_2, \cdots\cdots\cdots \alpha_n) \quad \ldots \quad (B)$$

On account of the one-one correspondences the number of ways of forming (B) is the same as the number of ways of forming (A) which is simply the number of ways of selecting $(n-1)$ distinct numbers all less than $(n+r)$ and ordering them in ascending magnitude, *viz.* ${}^{n+r-1}C_{n-1}$ or ${}^{n+r-1}C_r$

$$\therefore\ {}_nH_r = {}^{n+r-1}C_r.$$

Maharajah's College,
Mysore.

A. A. KRISHNASWAMI AYYANGAR.

IV On Vandermonde's Theorem

The following is a simple[†] proof of Vandermonde's theorem. It is seen here as a direct consequence of Leibnitz's theorem for the repeated differentiation of the product uv of two functions u and v of x.

Theorem :—

$$\sum_{s=0}^{r} {}_mC_{r-s}\ {}_nC_s = {}_{m+n}C_r$$

for all m and n and positive integral r.

Proof :—Applying Leibnitz's theorem with $u = x^m$ and $v = x^n$ we have

$$\sum_{s=0}^{s=r} {}_rC_s\ m(m-1)\cdots(m-r+s+1)x^{m-r+s}\ n(n-1)\cdots(n-s+1)x^{n-s}$$
$$=(m+n)(m+n-1)\cdots\cdots(m+n-r+1)\,x^{m+n-r}$$

On dividing both sides by x^{m+n-r},

$$\sum_{s=0}^{r} {}_rC_s\ \frac{m!}{(m-r+s)!}\ \frac{n!}{(n-s)!} = \frac{(m+n)!}{(m+n-r)!}$$

On dividing both sides by $r!$ and noting that

$${}_nC_r = \frac{n!}{(n-r)!\ r!}$$

we have the result.

[†] Cf :—Cayley's Proof, *Messenger of Mathematics* Vol. V.

Andhra University,
Waltair

K. SAMBASIVA RAO.

REVIEWS

ACTUALITES SCIENTIFIQUES ET INDUSTRIELLES (Hermann, Paris)

Colloque Consacré a la Théorie des Probabilités:

PREMIERE PARTIE:	*Conférences d'introduction*: Introduction par M. R. Wavre, Allocution par M. M. Frechet, Trois Conférences par M. M. Frechet, M. W. Heisenberg et M. G. Polya. Pp. 51. 11 fr. (1938) No. 734.
DEUXIEME PARTIE:	*Les Fondements du Calcul des Probabilités*: par M. M. P. Cantelli, W. Feller, M. Frechet, R. De Mises. J. F. Steffensen et A. Wald. Pp. 99. 21 fr. (1938). No. 735.
TROISIEME PARTIE:	*Les Sommes et les Fonctions de Variables Aléatoires*: par MM. H. Cramér, P. Lévy et R. De Misés. Pp. 68. 20 fr. (1938). No. 736.
QUATRIEME PARTIE:	*Le Principe Ergodique et les Probabilités en Chaîne*: par MM. E. Hopf, B. Hostinsky, O. Onicescu, V. Romanovsky Pp. 66. 18 fr. (1938) No. 737.
CINQUIEME PARTIE:	*Les Fonctions Aléatoires*: Par MM. S. Bernstein, E. Slutsky, A. Steinhaus. Pp. 73. 20 fr. (1938) No. 738.
SIXIEME PARTIE:	*Conceptions Diverses*: Par MM. B. de Finetti, V. Glivenko, G. Neyman. Pp. 57. 11 fr. (1938) No. 739.
SEPTIEME PARTIE:	*La Statistique Mathematique*: Par MM. E. Dodd, C. Jordan, N. Obrechkoff. Pp. 64. 18 fr. (1938) No. 740.

The Theory of Probabilities is a branch of mathematics which is rapidly coming to the forefront as an essential tool of research in several fields of knowledge as widely separated as Quantum Theory and Business Economics. Almost every mathematician is transforming himself into a probabilist and channels are opened out to make the highest achievements of pure mathematics available for rearing this new edifice. International Conferences are called in for mathematical co-operation in the new architecture. In our own country, the Indian Science Congress was compelled to recognise Mathematics as a separate section, by virtue of its association with Statistics. The year 1937 has witnessed two conferences, one in each hemisphere, on the subject of Probability. The earlier conference took place at the Graduate School of the United States Department of Agriculture where J. Neyman spoke about the modern view point on the classical Theory of Probability, his theories of confidence intervals and statistical estimation, and application to plant breeding, time-series analysis in economics and census by sampling. A few months later, in October 1937, there was a 'Colloquy' on 'Probabilities' organised by the University of Geneva. The original intention of the organisers was to have both 'Probabilities' and their 'applications'. But as the response to first invitations was full of probabilists, the idea of 'applications' was given up for the moment. The memoirs consecrated to the pure theory are of such diverse character and so numerous that even the eminent president Maurice Frechet despairs of presenting a complete resumé of the situation. As the organisers had given entire freedom in the selection of subjects, it is perhaps not illegitimate to infer that the subjects presented here indicate the direction of movement in current researches. Usually the proceedings of Geneva Conferences are published in L'Enseignement mathematique (edited by M. Fehr) and other Swiss reviews. But as these reviews cannot accommodate more than 300 pages of matter at a time, it was thought fit to have recourse to the house of Hermann & Co for a rapid and early publication of the papers on 'probability' discussed at the conference. This is the genesis of the eight fascicules devoted to this colloquy. Seven of these are noticed here.

The titles of the fascicules give only a rough indication of the contents, the discrepancies in placement being probably due to convenience of printing. The first part is the most heterogeneous and the third least, while the sixth part has a vague heading 'diverse conceptions'. The papers may be re-ordered thus:

1. Les Fondements du Calcul des Probabilites; by M. Frechet, W. Feller, R. de Mises, J. F. Steffensen, A. Wald and B. de Finetti.

2. Le Principe Ergodique et les Probabilites en Chaine, by G. Polya, E Hopf, B. Hostensky, O. Onicescu, V. Romanovsky and W. Heisenberg.

3. Les Sommes et les Fonctions de Variables Aléatoires by H. Cramer, P. Levy, R. de Mises.

4. Les Fonctions Aléatoires by S. Bernstein, E. Slutsky, A. Steinhaus, and V. Glivenko.

5. La Statistique Mathematique by G. Neyman, E. Dodd, C. Jordan, and N. Obrechkoff.

There are twenty-three papers in all, fully reported, one in English, three in German and the rest in French. The titles of the papers are given below (translated into English) in the order in which they appear in the fascicules.

M. Frechet: On the current principles in the evolution of recent researches on the Calculus of Probabilities.

G. Polya of Zurich: Random walk in a net-work of roads. (This was accompanied by a film demonstration).

W. Heisenberg of Leipzig: Probability notions in the Quantum theory of wave-fields.

P. Cantelli of Rome: On the definition of Contingent Variables (Only a short abstract appears here).

W. Feller of Stockholm: The axioms of the Calculus of Probabilities and their relation to experience.

M. Frechet of Sorbonne: Exposition and discussion of some recent researches on the fundamentals of the Calculus of Probabilities.

R. de Mises of Istanbul: Some remarks on the fundamentals of the Calculus of Probabilities.

J. F. Steffenson of Copenhagen: Frequency and Probability.

A. Wald of Vienna: On the validity of the notion of 'Collectives.'

P. Levy of Paris: The arithmetic of the laws of Probability and the finite products of the laws of Poisson.

R. de Mises: Generalisation of the classical limit theorems.

E. Hopf of Leipzig: The Statistical problem and its sequel in Classical Mechanics.

B. Hostinsky of Masaryk: Random fluctuations in the number of objects or points in a compartment.

O. Onicescu of Bucharest (Rumania): General theory of chains of complete connection (liaison).

V. Romanovsky of Tachkent: Some new problems in the theory of Markoff Chains.

S. Bernstein of Leningrad: Differential Stochastic Equations.

E. Slutsky of Moscow: On almost periodic random functions and the decomposition of stationary random functions into components.

H. Steinhaus of Lwow (Poland): Theory and applications of functions independent in the stochastic sense.

B. de Finetti of Trieste (Italy): On the condition of Partial Equivalence.

A. Kolmogoroff of Moscow: On random functions (laws of distribution in functional space) and their applications (This paper was not ready for publication here. It will be published in the series 'Exposes d' analyse generale' directed by M. Frechet in Actualites Scientifiques et Industrielles).

V. Glivenko of Moscow: On the law of large numbers in functional space.

J. Neyman of London: Statistical estimation treated as a classical problem of Probability.

E. Dodd of Texas, Austin (U. S. A.): Certain coefficients of regression or trend associated with largest likelihood.

C. Jordan of Budapest (Hungary): Criticism of Correlation from the probability point of view.

N. Obrechkoff of Sofia (Bulgaria): On the laws of Poisson, the series of Charlier and finite difference equations of the first order with constant coefficients.

From the above list of topics and their authors it is evident that we are entirely in a new world of probabilities little known to the English speaking people. We are more or less on Russian soil and its borders Scandinavia, Poland and the Balkan States, with Markoff's Chains elaborated by Onicescu and others, Kolmogoroff's functional spaces, Bernstein's differential stochastic equations, Mises's Collectives and Cramer's generalisation of the famous Laplace-Liapounoff limit-theorem. Two Germans speak on the new probability applications to Quantum theory and classical mechanics. The solitary voice of J. F. Steffensen champions forth the statistical definition of 'Probability' and does not see any reason for the several new theories of probability (each differing from the others in some detail adjusted to suit a particular pet view-point), as there is for the existence of one Geometry for Einstein and another for land-Surveyors.

There are two fundamental facts, which according to Henri Poincare characterise 'chance': (i) a very large number of small causes, each feeble in effect, acting simultaneously, and (ii) the existence of one or a small number of small causes, each producing a large effect. Gauss and Laplace have tackled mainly the first of the above two aspects, and Poincare had devised the method of arbitrary functions to deal with the latter. Poincare's method is extended by Hostinsky, Eberhard, Hopf, Copeland and de Mises. For a proper appreciation of these methods, it is necessary to consult the treatises of Borel, Castelnuvo, Khintchine, Kolmogoroff and Frechet. R. de Mises's lectures on 'Probability, Statistics and Truth' recently translated into English gives an excellent exposition of the theory of 'Collectives'. This theory has received further refinements from Popper, Reichenbach, Copeland, Wald and Ville. But S. Dumas and de Finetti who are both actuaries have not accepted the definition of de Mises. As an alternative to the 'Collective' theory, there is another characterised by Neyman as 'la definition classique modernisee', based on the modern theory of sets of points, sample space, and Lebesgue measure.

We are fairly familiar with Neyman's theory of statistical estimation as opposed to R. A. Fisher's, as they are accessible enough to English readers through the Statistical Research Memoirs of the Department of Statistics, University College, London and the American publication referred to in the earlier part of this review. This theory is adumbrated here with several new definitions, such as 'ensemble estimateur', 'ensembles d'acceptation' and 'ensembles estimateurs bien places' and theorems leading to the construction of 'intervals of confidence'. Neyman's paper may have therefore been included more justifiably under 'La Statisque Mathematique' than under the vague title 'Conceptions Diverses'. Among the 'statistical applications' which are, of course, very few we find, to our great relief, a paper in English by E. Dodd on the nature of the coefficient of regression expressed as a more general mean than the weighted arithmetic mean of the ratios of variables y_i x_i. The author derives several expressions with the help of R. A. Fisher's method of maximum likelihood applied to appropriate frequency distributions.

The notion of chains which was introduced in analysis by Markoff about three decades ago for the study of sequences of linked events appears now called on to play a fundamental role not only as a mathematical idea but as an instrument for the study of several natural phenomena. If $E_1, E_2, \cdots E_m$ be a finite aggregate of states in a system and p_{ik} the probability of passage from E_i to E_k, we have a simple chain. The infinite sequences of states $E_{i_1}, E_{i_2}, \cdots E_{i_n}, \cdots$ where i_n is simply one of the m given states will give rise to the infinite chain of probabilities $p_{i_1 i_2}, p_{i_2 i_3}, \cdots p_{i_{n-1} i_n} \cdots$ This scheme is applicable also to a continuous aggregate of states.

If $p_{ik}^{(n)}$ be the probability of passing by n successive changes from E_i to E_k, we get the recurrence formula

$$p_{ik}^{(n+1} = \sum_{s=1}^{m} p_{is}\ p_{sk}^{(n}$$

More generally,

$$p_{ik}^{\ n+l} = \sum_{s=1}^{m} p_{ik}^{(n)}\ p_{ik}^{\ l} \qquad \dots \quad (A)$$

Markoff has demonstrated that under sufficiently general circumstances, lim $p_{ik}^{(n)} = p_k$, i.e, the limiting probability of the system which attains the state of E_k is independent of the initial state. The 'ergodic principle' is based on this result. One generalised form of the chain is

$$p_{i_1},\ p_{i_1 i_2},\ p_{i_1 i_2 i_3},\ \cdots$$

where the probability of any change depends on that of all of its predecessors. Such chains are to be found in the phenomena of heredity.

Onicescu constructs what he calls 'les chains à liasons completes' which has its statistical analogue in nature in the spread of a contagious disease like tuberculosis which is also hereditary. This chain obeys a composition law which generalises (A) in the form

$$p_s^{\ n}\ (x) = \sum_{w=1}^{m+1} x_w\ p_s^{(n-1)}\ [f_{w_1}\ (x),\ f_{u_2}\ (x),\ \cdots\ f_{um}\ (x)].$$

Regarding the fundamental formula (A), Onicescu remarks that differential equations, equations of propagation of waves, matrices in Hilbert's space and chains of Markoff are all governed by the same group based on (A). This puts in evidence the rigorous determinism which dominates the propagation of states in a material system.

We are tempted to quote Laplace's prophetic words: 'If one considers the analytic methods to which this theory (of probability) has given birth, the truth of the principles on which it is based, the fine and delicate logic which is demanded by their employment in the solution of problems, the matters of public importance which are dependent on them, and the extension which it has received and which it can yet receive by its applications to the most important questions of natural philosophy and moral science .. ..
one sees that there is no science more worthy of our meditations, and that it would be most useful to introduce it into our system of public instruction.' We have nothing to add to Laplace's recommendation to introduce the subject seriously in our Honours Courses and post-graduate studies. The publications like those before us will be an unfailing source of inspiration to the interrogators of 'L'urne.'

We wish the *Makers of Probability* all success.

A. A. K.

ANNOUNCEMENTS AND NEWS

The following gentlemen have been elected members of the Society :—

- V. P. Venkatachari Esq., M.A., Oosmania University, Hyderabad, (Deccan).
- H. Subramani Iyer Esq., M.A., Ph D., Travancore University, Trivandrum.
- P. C Mahalanobis Esq., M.A., B.SC., I. E. S., Head of the Statistical Laboratory, Presidency College, Calcutta.

The Librarian wishes to announce that owing to the International situation (for which he hints he is not to be blamed) the number of available current mathematical periodicals has diminished very considerably. He invites the cooperation of all members in preventing stagnation in the circulatory system.

Dr. Hansraj Gupta of Government College, Hoshiarpur, is collecting notes relating to the lives and work of mathematicians engaged in research in India with a view to the publication of this material later. He requests their cooperation in the work by sending him 2 copies of their photographs, a short sketch of their lives, a complete list of their published papers with the fullest details and short summaries of the most important papers.

Sri S. Minakshisundaram, M.A. has been appointed Research Assistant in Topology in the Madras University to help Dr. Vaidyanathaswami in giving guidance to research students.

SOLUTIONS TO QUESTIONS

Question 1597

(R. **Ramanujan**): Given four line segments measuring 4, 9, 12 and 21 units respectively, find geometrically the values of x and y satisfying the equations

$$\frac{xy\,(x+y+8)}{21\,(x+y)} - \frac{9\,(x+y)}{25\,(x+y-8)} = (x^2-y^2)^2 - 63\,(2x^2+2y^2-91) = 0.$$

Solution by the Proposer

The second equation simplifies into

$$x^4+y^4+8^4-2x^2y^2-2x^2\cdot 8^2-2y^2\cdot 8^2+64\times 27=0$$

which can be written

$$\left\{\frac{x+y+8}{2}\cdot\frac{x-y+8}{2}\cdot\frac{-x+y+8}{2}\cdot\frac{x+y-8}{2}\right\}^{\frac{1}{2}} = \sqrt{108} \qquad \ldots (1)$$

the first equation becomes

$$\left\{xy\left[1-\frac{8^2}{(x+y)^2}\right]\right\}^{\frac{1}{2}} = \frac{\sqrt{(189)}}{5} \qquad \ldots (2)$$

If x, y, 8 are the sides of triangle, the left side of (1) gives the area and that of (2) gives the length of the internal bisector of the vertical angle contained by the sides x and y. If a triangle with sides x, y, 8 and the left side of (2) for the length of the external bisector be drawn then $(x, -y)$ and $(-x, y)$ will be solutions of the given equations. These considerations give the following geometric method of solving the given equations:

Construct a triangle satisfying the following condition:—(i) base = 8 units (ii) area = $\sqrt{(108)}$ units (iii) the length of the internal bisector of the vertical angle equals $\frac{\sqrt{(189)}}{5}$ units. Let x_1 and y_1 be the lengths of the remaining sides. Similarly construct another triangle satisfying (i), (ii) and (iii), the external bisector of the vertical angle being of length $\frac{\sqrt{(189)}}{5}$ units. Let x_2 and y_2 be the lengths of the remaining two sides. Then the four solutions of the equations are (x_1, y_1), and $(-x_1, -y_1)$ (x_2, y_2) and $(-x_2, y_2)$. That the above construction can be carried out by using the 4 segments of given lengths and a pair of compasses is a matter of detail which can be verified.

Question 1668

(ALFRED MÖSSNER):—We have

$$17^n + 17^n = (-3)^n + 5^n + 12^n + 20^n \qquad (n = 1, 2, 3)$$

Is the equation $A_1^n + A_2^{2n} = B_1^n + B_2^n + B_3^n \qquad (n = 1, 2, 3)$ also solvable when $A_1 \neq A_2$? What is the general solution?

(ii) We have $7^n + 7^n = 3^n + 5^n + 8^n \qquad (n = 2, 4)$

Is the equation $E_1^n + E_2^n = F_1^n + F_2^n + F_3^n \qquad (n = 2, 4)$ also solvable for $E_1 \neq E_2$?

Remarks by A. Weiland

The following is a solution when n takes only the values 1 and 3. The identity

$$(t^2 - 2t - 1)^k + (t - 1)^{2k} = 2^k + (t^2 - 3t)^k + (t^2 - t - 2)^k$$

is easily verified to be true for $k = 1$ and 3. For $t > 3$ all the expressions involved are positive. Thus we get for

$$t = 4, \quad \left.\begin{array}{l} 7^3 + 9^3 = 2^3 + 4^3 + 10^3 \\ 7 + 9 = 2 + 4 + 10. \end{array}\right\};$$

$$t = 5, \quad \left.\begin{array}{l} 14^3 + 16^3 = 2^3 + 10^3 + 18^3 \\ 14 + 16 = 2 + 10 + 18. \end{array}\right\};$$

$$t = 6, \quad \left.\begin{array}{l} 23^3 + 25^3 = 2^3 + 18^3 + 28^3 \\ 23 + 15 - 2 + 18 + 28. \end{array}\right\};$$

$$t = 7, \quad \left.\begin{array}{l} 34^3 + 36^3 = 2^3 + 28^3 + 40^3 \\ 34 + 36 = 2 + 28 + 40. \end{array}\right\}.$$

Question 1775

(A. A. KRISHNASWAMY AYYANGAR).—Prove the following results

$$\frac{\pi}{\sqrt{12}} = 1 - \frac{1}{3^2} + \frac{1}{5 \cdot 3^2} - \frac{1}{7 \cdot 3^3} + \cdots\cdots$$

$$\frac{\pi}{16} = \frac{1}{1^5 + 4 \cdot 1} - \frac{1}{3^5 + 4 \cdot 3} + \frac{1}{5^6 + 4 \cdot 5} - \cdots\cdots$$

$$\frac{\pi - 3}{4} = \frac{1}{3^3 - 3} - \frac{1}{5^3 - 5} + \frac{1}{7^3 - 7} - \cdots\cdots$$

$$\frac{4 - \pi}{8} = \frac{1}{4^2 - 1} + \frac{1}{8^2 - 1} + \frac{1}{12^2 - 1} + \cdots\cdots$$

$$\frac{\pi - 3}{6} = \frac{1}{(2 \cdot 2^2 - 1)^2 - 2^2} + \frac{1}{(2 \cdot 4^2 - 1)^2 - 4^2} + \frac{1}{(2 \cdot 6^2 - 1)^2 - 6^2} + \cdots$$

Solution by C. B. Rathie.

1. $$1-\frac{1}{3^2}+\frac{1}{5\cdot 3^2}-\frac{1}{7\cdot 3^3}$$

$$=\sqrt{3}\left\{\frac{1}{\sqrt{3}}-\frac{1}{3}\cdot\left(\frac{1}{\sqrt{3}}\right)^3+\frac{1}{5}\cdot\left(\frac{1}{\sqrt{3}}\right)^5-\frac{1}{7}\left(\frac{1}{\sqrt{3}}\right)^7+\cdots\cdots\right\}$$

$=\sqrt{3}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$, by applying Gregory's Series,

$$=\sqrt{3}\cdot\frac{\pi}{6}=\frac{\pi}{\sqrt{12}}.$$

The above series is identical with that of Ex. 6, page 112, of Loney's Trigonometry Part II.

2. $$\frac{1}{1^5+4\cdot 1}-\frac{1}{3^5+4\cdot 3}+\frac{1}{5^5+4\cdot 5}-\cdots\cdots$$

We have $\dfrac{1}{x^5+4x}=\dfrac{1}{4}\left\{\dfrac{1}{x}-\dfrac{x^3}{x^4+4}\right\}$.

It is easily seen that $\dfrac{x^3}{x^4+4}=\dfrac{1}{2}\left\{\dfrac{x-1}{(x-1)^2+1}+\dfrac{x+1}{(x+1)^2+1}\right\}$.

Therefore, $\dfrac{1}{x^5+4x}=\dfrac{1}{4}\cdot\dfrac{1}{x}-\dfrac{1}{8}\left\{\dfrac{x-1}{(x-1)^2+1}+\dfrac{x+1}{(x+1)^2+1}\right\}$.

Putting $x=1, 3, 5, \cdots\cdots$, we get

$$\frac{1}{1^5+4\cdot 1}=\frac{1}{4}\cdot\frac{1}{1}-\frac{1}{8}\left\{0+\frac{2}{2^2+1}\right\},$$

$$-\frac{1}{3^5+4\cdot 3}=-\frac{1}{4}\cdot\frac{1}{3}+\frac{1}{8}\left\{\frac{2}{2^2+1}+\frac{4}{4^2+1}\right\},$$

$$\frac{1}{5^5+4\cdot 5}=\frac{1}{4}\cdot\frac{1}{5}-\frac{1}{8}\left\{\frac{4}{4^2+1}+\frac{6}{6^2+1}\right\},$$

Hence, on adding, the given series

$$=\frac{1}{4}\left\{1-\frac{1}{3}+\frac{1}{5}-\cdots\cdots\right\}+0=\frac{\pi}{16}.$$

3. $$\frac{1}{3^3-3}-\frac{1}{5^3-5}+\frac{1}{7^3-7}-\cdots\cdots$$

$$=\frac{1}{3}\cdot\frac{1}{3^2-1}-\frac{1}{5}\cdot\frac{1}{5^2-1}+\frac{1}{7}\cdot\frac{1}{7^2-1}-\cdots\cdots$$

$$=\frac{1}{3}\cdot\frac{1}{2\cdot 4}-\frac{1}{5}\cdot\frac{1}{4\cdot 6}+\frac{1}{7}\cdot\frac{1}{6\cdot 8}-\cdots\cdots$$

$$=\frac{1}{2}\left\{\frac{1}{3}\left(\frac{1}{2}-\frac{1}{4}\right)-\frac{1}{5}\left(\frac{1}{4}-\frac{1}{6}\right)+\frac{1}{7}\left(\frac{1}{6}-\frac{1}{8}\right)-\cdots\cdots\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{2\cdot3} - \frac{1}{3\cdot4} - \frac{1}{4\cdot5} + \frac{1}{5\cdot6} + \frac{1}{6\cdot7} - \frac{1}{7\cdot8} - \cdots\cdots\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{2} - \frac{1}{3} - \frac{1}{3} + \frac{1}{4} - \frac{1}{4} + \frac{1}{5} + \frac{1}{5} - \frac{1}{6} + \frac{1}{6} - \frac{1}{7} - \frac{1}{7} + \frac{1}{8} - \cdots\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{2} - 2\left(\frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \cdots\right)\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{2} - 2(1 - \tan^{-1} 1)\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{2} - 2 + \frac{\pi}{2}\right\} = \frac{\pi - 3}{4}.$$

4. $$\frac{1}{4^2 - 1} + \frac{1}{8^2 - 1} + \frac{1}{12^2 - 1} + \cdots$$

$$= \frac{1}{3\cdot5} + \frac{1}{7\cdot9} + \frac{1}{11\cdot13} + \cdots$$

$$= \frac{1}{2}\left\{\frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \frac{1}{11} - \frac{1}{13} + \cdots\right\}$$

$$= \frac{1}{2}\{1 - \tan^{-1} 1\} = \frac{1}{2}\left(1 - \frac{\pi}{4}\right) = \frac{4 - \pi}{8}.$$

5. $$\frac{1}{(2\cdot2^2 - 1)^2 - 2^2} + \frac{1}{(2\cdot4^2 - 1)^2 - 4^2} + \frac{1}{(2\cdot6^2 - 1)^2 - 6^2} + \cdots$$

Now $$\frac{1}{(2x - 1)^2 - x} = \frac{1}{4x^2 - 5x + 1} = \frac{1}{(x - 1)(4x - 1)}$$

$$= \frac{1}{3}\left\{\frac{1}{x - 1} - \frac{4}{4x - 1}\right\}.$$

Putting $x = 2^2, 4^2, 6^2, \cdots\cdots$

$$\frac{1}{(2\cdot2^2 - 1)^2 - 2^2} = \frac{1}{3}\left\{\frac{1}{2^2 - 1} - \frac{4}{4\cdot2^2 - 1}\right\} = \frac{1}{3}\left\{\frac{1}{1\cdot3} - \frac{4}{4^2 - 1}\right\},$$

$$\frac{1}{(2\cdot4^2 - 1)^2 - 4^2} = \frac{1}{3}\left\{\frac{1}{4^2 - 1} - \frac{4}{4\cdot4^2 - 1}\right\} = \frac{1}{3}\left\{\frac{1}{3\cdot5} - \frac{4}{8^2 - 1}\right\},$$

$$\frac{1}{(2\cdot6^2 - 1)^2 - 6^2} = \frac{1}{3}\left\{\frac{1}{6^2 - 1} - \frac{4}{4\cdot6^2 - 1}\right\} = \frac{1}{3}\left\{\frac{1}{5\cdot7} - \frac{4}{12^2 - 1}\right\},$$

. . . .

Hence on adding, the series becomes

$$= \frac{1}{3}\left\{\frac{1}{1\cdot3} + \frac{1}{3\cdot5} + \frac{1}{5\cdot7} + \cdots\right\} - \frac{4}{3}\left(\frac{1}{4^2 - 1} + \frac{1}{8^2 - 1} + \frac{1}{12^2 - 1}\right)\cdots$$

$$= \frac{1}{3}\left\{\frac{1}{2}\left(1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \frac{1}{5} - \frac{1}{7} + \cdots\right)\right\} - \frac{4}{3}\cdot\frac{4 - \pi}{8},$$ by result 4.

$$= \frac{1}{6} - \frac{4 - \pi}{6} = \frac{\pi - 3}{6}.$$

QUESTIONS FOR SOLUTION

1788. (A. E. PALKAR). Show that the number $2^{9061452}$ leaves the remainder 137438429185 when divided by $(524287)^2$.

1789. (P. KESAVA MENON). If (r, n) denotes the greatest common measure of r and n, and $\phi(n)$ is the number of integers less than n and prime to it and if $n = p_1^{a_1} \, p_2^{a_2} \cdots p_t^{a_t}$, the p's being different prime factors, show that

$$\sum_{1 \leqslant r_1, r_2, \cdots r_s \leqslant n} (r_1 r_2 \cdots r_s, \, n) = \phi^s(n) \prod_{q=1}^{t} \binom{s+a_q}{r+a_q} (p_q - 1)^{-r}$$

$$= n^s \prod_{q=1}^{t} \frac{a_q(a_q+1) \cdots (a_q+s)}{s\,!} \prod_{q=1}^{t} \sum_{r=0}^{s} (-1)^r \frac{\binom{s}{r}}{p_q^r(a_q+r)} .$$

1790. (A. A. KRISHNASWAMI AYYANGAR). Elucidate from considerations of symmetry (without induction) that the symmetric function

$$\sum \alpha_1^{p_1} \, \alpha_2^{p_2} \cdots \alpha_n^{p_n}$$

can be expressed in terms of power-sums

$$S_p \equiv \alpha_1^p + \alpha_2^p + \cdots + \alpha_r^p \qquad (r > n)$$

in the form

$$\sum A_{p'q'} \cdots S_p \, S_{q'} \cdots\cdots\cdots$$

where $\quad p' + q' + \cdots = p_1 + p_2 + \cdots + p_n$

and $\quad p', q', \cdots$ are ths sums of different sets of the indices $p_1, p_2, \cdots p_n$, the summation extending over all the possible values of $p', q', \cdots$ under the circumstances.

Prove that $A_{p_1 q'} \cdots$ depends only on the number of the integers $p', q', \cdots$ and the number of indices (p_1, p_2, p_n) that go to form $p', q', \cdots$.

Hence deduce the value of $A_{p_1 q'} \cdots$ by putting

$$p_1 = p_2 = \cdots = p_n = 1.$$

1791. (P. KESAVA MENON). If through any point O three secants are drawn to meet a conic in P, P^1; Q, Q^1; R, R^1 respectively then

$$\frac{\Delta \, PQR}{OP \cdot OQ \cdot OR} + \frac{\Delta \, P^1Q^1R^1}{OP^1 \cdot OQ^1 O \cdot R^1} = 0.$$

1792. (N. A. COURT, Oklahoma, U. S. A.). Find a sphere such that the polar planes with respect to this sphere of two given points shall pass respectively through two given lines.

THE MATHEMATICS STUDENT

Volume VIII] SEPTEMBER 1940 [Number 3

RENE DESCARTES*

BY

A. A. KRISHNASWAMI AYYANGAR, *Maharaja's College, Mysore.*

Rene Descartes was born on March 31, 1596 at La Haye near Tours, the last child of his mother who died a few days after the delivery. To give an idea of the intellectual period in which he was born, the ages of his contemporaries then may be mentioned, Vieta 56, Napier 46, Harriot 36, Briggs 35, Shakespeare and Galileo 32 Kepler 25, Oughtred 21, Mersenne and Desargues 8 and 3 respectively, while Fermat (b. 1601), Roberval (b. 1602), Milton (b. 1608), Wallis (b. 1616), Pascal (b. 1623), Huyghens (b. 1629), Newton (b. 1642) and Leibnitz (b. 1646) were all born within the lifetime of Descartes. Rene's father provided his inquisitive son an excellent nurse who was often worried by him to substantiate her statements about heaven. These early fundamental doubts inspired Rene s life-work and destined him to become a great philosopher and a greater mathematician.

On account of his delicate health he was put to school late. His father wisely put him under the care of the good rector Father Charlet of the Jesuit College at La Fleche. Finding that the boy required more rest than the normal boys of his age, the good rector allowed him to lie in bed as late as he wished in the mornings, and to join his fellows in the class-room only when he felt like doing so. Rene was very grateful to the Father for this kindness and did much good work in silent meditations in bed. In his later days he recollected that the happiest period of his life were those quiet mornings in bed at La Fleche. In 1647, when he visited Pascal, another invalid mathematician of exceptional powers, he advised him not to allow himself to be disturbed in the mornings from his bed in order to preserve his health and mathematics. Pascal did not heed this advice and died early. At La Fleche Rene made some life-long friends. The most notable of them was Mersenne, later

---

* This paper was read before the Philosophical Association, Maharaja's College, Mysore.

Father, the famous scientific amateur who made it his business to befriend and correspond with all the French mathematicians of his time and many foreign contemporaries also. Rene emerged from his college in his seventeenth year with all the equipment that the educational tradition of those days could give him.

His scepticism increased with his studies. He found the humanities that he studied were bereft of human significance and the logic of the schoolmen of the middle ages as barren as a mule for any creative human purpose. Disgusted with his studies, he resolved to see the world for himself, life in the raw. As he was financially well off to do as he pleased, he proceeded to Paris to enjoy life. He came into contact with the great mathematical thinkers of his day. He also gambled with enthusiasm and some success. The company of gamesters, however, soon tired him and he returned to his mathematics. But his hare-brained friends would not leave him. To get a little peace he went to war. He served the army in the Netherlands, Germany and Austria. But as there were no serious wars, the peaceful life of the camp also bored him though it gave him some leisure to dream about his mathematics and philosophy. His first idea of analytical geometry and philosophy came to him in some dreams that he had on the night of November 10, 1619, at Neuberg where his army was resting in its winter-quarters on the Danube. These dreams imply a sub-conscious working out of the possibilities of exploring natural phenomena by the aid of mathematics.

He took leave of his military activities in 1621. He returned to his father's home, sold away his portion of paternal property and invested the proceeds in such a manner as to fetch him an annual income of about five hundred pounds in modern currency. He then went away on a pilgrimage to Italy, and without being able to meet Galileo, the father of modern science, came back to Paris, the Paris of Cardinal Richelieu in 1625. Galileo's astronomical researches with his crude telescope had set all the natural philosophers of Europe grinding lenses. With his mathematical friends Mersenne and Mydorge, Rene became deeply interested in the theory of refraction of light and in investigating the best shape for glasses in optical instruments. As his hands were engaged in grinding glasses, his mind was equally engaged with reflections on the nature of man soul and God.

In spite of his lofty thoughts, Descartes had a weakness for gaudy and fashionable dress and went about in costly taffeta with a broad-brimmed, ostrich-plumed hat. He had his own share of women in the world and had a daughter by one. The child's early death affected him deeply. He never married, and saved himself from designing Dutch widows by the announcement of his preference of truth to beauty.

The three years 1625–28 were the happiest of Descartes' life. But his increasing reputation brought him a horde of fashionable friends. He once more escaped his friends by taking military service, this time with the King of France at the siege of La Rochelle, where he met Cardinal Richelieu and each was impressed by the other. Till 1628, he had published nothing but on being acquainted with two good Jesuit cardinals who passionately loved the sciences and listened with admiration to his philosophy, he was too good a Catholic to resist their appeal to share his discoveries with the world, thereby avoiding the penalty of hell-fire which was threatened to be his lot otherwise.

The next twenty years of Descartes' life were spent in wandering all over Holland, methodically carrying on scientific and philosophic correspondence with all the leading intellects of Europe and occupying himself with all things that came in his way. Anatomy, Chemistry, Embryology, Medicine, Meteorology, Musical Theory, and Optics (in alphabetical order) were all grist to his mill. When he visited England he was introduced to the behaviour of the magnetic needle and forthwith magnetism was also included in his study. Theological speculations did not fail to attract his attention.

In 1634, Descartes was about to publish his physical theory of the Universe entitled Le Monde and was awaiting to see Galileo's latest book on science to put the final touches to his own. But instead of receiving a copy of the book he heard the shocking news that Galileo had been given up to the Inquisition, which forced him to abjure on his knees the Copernican doctrine that the earth moves round the sun. This is the tragedy of mathematical truths or for that matter any untimely truth which is opposed to vested interests. In the ecclesiastical imagination, Galileo's assertion of the earth's motion was tantamount to Galileo himself setting the earth in motion and bringing on some disastrous consequences like the earthquake. What business, indeed, had Galileo in prying into Nature's secrets and much more in publishing them? Nature herself was wroth with Galileo and connived in his punishment. Galileo's punishment proved such a deterrent to Descartes that he was afraid to publish not only his Le Monde but his other projected works until some friends induced him to do so with sufficient guarantees of safety. In fact Cardinal Richelieu gave Descartes the privilege of publishing either in France or abroad anything he cared to write. But there were many Protestant theologians who savagely condemned Descartes' work as atheistic and dangerous to the State.

Descartes next devoted himself to a treatise on Universal Science. This was published in Leiden in 1637 under the title 'Discours de la Methode pour bien conduire sa raison et chercher la verite dans les sciences' accompanied by three appendices, La Dioptrique, Les Meteores, La Geometrie, of which the last proved to be of epoch-making importance. In 1641, he published his 'Meditationes' explaining at length the philosophy in his Discourse. The two books attracted the notice of the Princess Elizabeth (a grand-daughter of James I of England) who was a young prodigy of learning then living in her exile with her mother in the Hague, which unfortunately adjoined the little village where Descartes lived in his philosophic seclusion. An interview was arranged between the princess and the philosopher resulting in the latter ultimately consenting to be her tutor. Elizabeth took to Descartes' analytic methods with great enthusiasm. Even after she left Holland, she continued to be in touch with the philosopher until the day of his death.

In 1644 Descartes issued the *Principia Philosophia* the greater part of which was devoted to the physical sciences, especially the laws of motion and the theory of vortices. In the vortex-theory, universal motion is assumed to produce a number of whirlpools. The sun was supposed to be the centre of an immense vortex of matter in which the planets float and move round like straws in a whirlpool. Each planet again is the centre of a secondary vortex carrying the satellites and producing variations of density in the primary vortex causing elliptic instead of circular motion. The theory though crude and badly torn to pieces in the Principia of Newton yet deserves some credit as an attempt to describe the phenomena of the universe in terms of mechanical laws obtaining on the earth. This theory continued to be adopted only in France for about a century until in 1738 the Newtonian theory of gravitation ultimately superseded it mainly through the witty raillery of Voltaire.

In 1646 Descartes was having his quiet life in Egmond, meditating, gardening, and carrying on scientific correspondence with the intellectual lights of Europe. He was now fifty, world-famous and indeed more famous than he ever cared to be. In 1647 he received a pension from the French Government in honour of his discoveries. But he was not left in peace to continue his great work. Queen Christine of Sweden had come to know of his existence. Like Elizabeth she had also a great passion for knowledge and wanted Descartes to be her instructor. Descartes was too snobbish to refuse the queen's request. The result was that he lost his peace and quiet morning hours of meditation in bed. Crawling out of bed at some ungodly hour in the dark, he had to climb into the carriage sent to 'collect' him and reach the queen's icy library punctually at 5 A.M. to give his lessons. He was also asked to plan out a Royal Swedish Academy of Sciences: he completed a few days before his death the statutes for this academy. Descartes could not get out of this work unless he used some rude words to his pupil, which he dared not. So he suffered until death got him release. He contracted inflammation of lungs, a disease caught from his host when nursing him. He died on February 11, 1650 aged 54. In 1667, the bones of Descartes were removed to France and re-entombed in what is now, the Pantheon. As Jacobi commented, it is often more convenient to possess the ashes of a man than the man himself. Descartes had an indiscreet temper which alienated him from his brilliant compeers and countrymen.

The collected works of Descartes first appeared in Latin at Amsterdam in nine volumes in 1690-1701. The French editions came much later. The latest edition by Charles Adam and Paul Tannery came out in nine volumes between 1897 and 1904.

Descartes was one of those universal personalities difficult to classify. He is claimed at the same time by philosophers, physicists and mathematicians as their man, perhaps with equal justification. He was one of the giants selected by Newton as his pedestal to enable him to see further. Descartes was one of those few men to whom it is given to renovate a whole department of human thought. Just as the age of physical science may be said to be inaugurated by Galileo, the age of modern mathematics may be said to begin with Descartes.

The modern algebraic notation is entirely due to him. He discovered some beautiful theorems in the Theory of Equations concerning the number and situations of the real roots, which are receiving their extensions even to-day. He gave a new solution of the equation of the fourth degree. It is interesting to remark that Bhaskara's solution (1150 A. D.) of a numerical equation of the fourth degree suggests a method leading to a cubic which is no other than Descartes' auxiliary equation with a slight modification. †

Descartes' greatest achievement lay in geometry. The synthetic methods of the Greeks did not offer any general tool for geometric research. Every problem was a jigsaw puzzle to be attacked by *ad hoc* special methods. Descartes' method introduced a new mathematical chakshus or vision with its parallel in simplicity and comprehensiveness only in the decimal notation which has pointed out a way of managing numbers in an infinitely ascending or descending progression. Changing the metaphor, we may say that geometry and algebra existed in different endogamic groups until Descartes celebrated the first marriage of the geometrical Eve and the algebraic Adam, himself acting as their first official priest, blessing their union with a public sermon. The result is 'analytical geometry'. 'The old-maidishness of Plato's rigidly corseted conception of what the Muse of Geometry should be,' disappeared with the Cartesian touch. Descartes not only released geometry from Platonic bondage but also gave her a pair of wings in the form of two co-ordinates with the help of which she can take flight into new worlds. The graphs which decorate the social sciences to-day are the direct descendants of this cartesian social reform.

The idea underlying analytical geometry is very simple. We all know that we can easily fix up a house if we know the name of the street in which the house is situated and the number of the house in the street. These two bits of information may be called the co-ordinates of the house. Even so, the position of a point in a plane may be uniquely fixed by two numbers, one giving the distance north or south (y) of a horizontal line in the plane, and the other the distance east or west (x) of a vertical line in the same plane. The point can now be represented by these two numbers which we call x, y co-ordinates. Suppose now the point begins to roam over the plane not in any random manner but according to a certain law, say, parallel to the horizontal line; then we say its y-co-ordinate is constant and write $y=k$. Again let it move such that its distance from the common point of the horizontal and vertical lines is constant; then we say its equation is $x^2+y^2=k^2$ while we know the path is a circle. Therefore $x^2+y^2=k^2$ represents a circle. Conversely from the equation we can pass on to the curve by finding x, y pairs satisfying the equation and locating them as points in a plane. If these points are sufficiently near they can be joined to obtain a curve. If we have two circles in the plane, we can write their corresponding equations, solve them and get number-pairs which tell us whether the circles cut each other, or not. Thus even without actually seeing the circles, we can work out their behaviour. Again, suppose a point chooses to move so as to keep its distance from a given point always equal to its distance from a given straight line, not passing through the point, what is its career like? The equation of the career can be readily written down by our new method and provides a comparison with that of the path of a projectile thrown in air. The similarity of the two equations shows that the two curves must be the same. The latter curve is known experimentally and therefore the former is laid bare. In this way, by a reduction of all curves to a common notation the solution of one problem would at once entail the solution of similar problems. In the language of Descartes the problem of curves is solved by a reduction to a problem of straight lines. In other words every geometric problem can be set to its algebraic expression and this implies that every improvement or discovery in Algebra is susceptible of application to geometry and vice-versa. When geometry and algebra thus get intimately inter-twined, it becomes impossible to improve the one without improving the other also. Henceforth we are released from the cobweb of lines in geometry and can write out equations more neatly and less confusingly than drawing lines which often give us misleading and false suggestions. This is another advantage in manipulating equations instead of manipulating lines. We would not be diverted by random configurations.

---

† Vide *Current Science*, Vol. VII. No. 4 p. 179 (October, 1938.)

The above epoch-making method which I may call Descartes *monism* in mathematics, is elaborated in his La Geometrie an appendix of 116 pages to his Discours published in 1637. This work was made purposely obscure for fear that too much lucidity will make jealous people advertise it as trivial. It was appreciated more in the youthful universities of Holland than in the ecclesiastically controlled French universities. Van Schooten's Latin edition of La Geometrie as well as Wallis's treatise on Conic Sections popularised the analytic method among mathematicians. The result was a crop of discussions on various curves, their tangent and normal properties, their centre of gravity, and volumes of solids obtained by revolving curves. One of the important curves is the Cycloid, called the Helen of geometers on account of its beautiful properties and the controversies which their discovery engendered. It is the curve which forms beautiful arches. It is also the curve on which a body sliding from rest under gravity will reach the lowest point in the same time from whatever point it starts. Descartes and Fermat succeeded in drawing tangents to this curve. As a challenge to Fermat, Descartes proposed his Folium for the construction of tangents. The correct shape of this curve was discovered in 1692 by C. Huyghens, the prince of philosophers in Holland, whose greatness Descartes had predicted. The famous quartic ovals invented by Descartes in connection with his inquiries about the best forms of converging lenses are now studied in connection with elliptic functions.

The analytical geometry was a worthy and a necessary precursor of the calculus of Newton and Leibnitz. Analytical geometry and Calculus are the two wonderful machines in which we have only to put certain functions or equations in order to make the result come out. It is no idle boast of Descartes that he had so far surpassed all geometry before him as Cicero's rhetoric surpassed the A, B, C. Descartes was not a great reader. Erudition was not his forte. Yet he was able to unfold a great science to the world, for he believed that the mind was not for the sake of knowledge but knowledge for the sake of the mind, and worked accordingly.

Ninety-nine years after Descartes had set the analytical ball rolling, Euler took it up for Mechanics. The glorious infection spread to Optics in 1827, when William Rowan Hamilton the greatest scientific man of Ireland constructed his algebraic theory of systems of rays, reducing all his researches to the story of a single function. Mathematical formulae seem to have an intelligence of their own showing the way as it were to their discoverers to seek hidden treasures in unknown directions in undreamt-of regions.

REFERENCES

1. *Men of Mathematics*, by E. T. Bell (1937).
2. *Encyclopedia Brittannica*, 11th Edition.
3. *History of Mathematics*, by F. Cajori.
4. *A short account of the History of Mathematics*, by W. W. R. Ball.

ENUMERATION OF FINITE GROUPS

BY

G. A. MILLER. *University of Illinois, Urbana, Illinois, U.S.A.*

The three principal kinds of enumerations of groups of finite order which have been published are the permutation groups which can be represented on a given small number of letters, the abstract groups whose orders do not exceed a given small number, and thirdly those which separately contain a given small number of subgroups. In all of these cases the enumerations soon led to difficulties which made further progress along the particular line seem impractical, but the groups which have been enumerated throw much light on the nature of the general subject of group theory. These enumerations should therefore be regarded as convenient means for securing an insight into important properties of groups rather than as opening lines of investigation which should be pursued persistently along the direct lines to which they relate even if such progress seems to have no limit.

The enumeration of permutation groups was started at the beginning of the serious study of group theory as an autonomous subject of mathematics. Some of the earliest writers thereon including A. L. Cauchy (1789-1857) and E. Methieu (1835-1890), confined their attention in these enumerations to the possible orders of the permutation groups. This is due to the fact that in the applications to the solutions of algebraic equations it is especially interesting to know whether rational functions of n variables exist which have a given number of formal values when these variables are permuted in every possible manner. The fact that functions which have the same number of values may give rise to different equations of the same degree appeared at first to be of secondary importance. Most of the later writers considered however the determination also of all the possible groups of the same order, but even for the comparatively small degree 12 no one has as yet published all the possible intransitive permutation groups which exist.

The enumeration of the possible abstract groups of low orders has been carried somewhat further. In volume 175, page 252 (1936), of the *Journal für die reine und angewandte Mathematik* the enumeration of all the abstract groups whose orders do not exceed 200 is proposed as a prize problem. It is here stated that the solvable groups whose orders do not exceed 96 have been determined, while the insolvable groups can easily be determined even for a much higher limit of orders. On account of the much greater number of solvable groups than of insolvable groups for small numerical orders it is the former which have given the greater trouble in the published enumerations, which may be said to have been inaugurated by A. Cayley (1821-1895), who published occasional papers on the subject of group theory during about 40 years extending from 1854 to 1895 and contributed both to abstract groups and also to permutation groups.

The third of the three kinds of group enumerations to which we referred above is the most recent and appeared in various parts of the *Proceedings of the National Academy of Sciences*, beginning with volume 25, page 367 (1939) under the title "Groups which have a small number of subgroups". From the standpoint of abstract group theory this is the most natural type of enumerations since the earliest and most efficient method to study abstract groups is by means of their subgroups, and the theorem known as Lagranges theorem which affirms that the order of a group is divisible by the order of each of its subgroups may reasonably be regarded as the most fundamental theorem of the entire subject of group theory. It is therefore inaccurate to say that Sylow's theorem was the first to be proved among a set of closely related theorems on the structure of finite groups, as is stated in the *Encyclopedia Brittannica*, volume 10, page 914 (1938). Unfortunately a large number of other mathematical errors appear in this edition of this widely used encyclopedia. Cf. *Science*, December 1, 1939, page 512.

A group G which is not the identity contains at least two subgroups, viz., the identity and the group itself. These two subgroups are called *improper* subgroups of G while each of its other subgroups is called a *proper* subgroup of G, if the order of G exceeds 2. The identity is known as an actual subgroup of G so that G has always one more actual subgroups than proper subgroups, if it is not the identity. When G is the cyclic group of order p^m, p being an arbitrary prime number, then G contains exactly $m-1$ proper subgroups. It is not difficult to prove the following theorem relating to groups which contain a given number of subgroups: *A necessary and sufficient condition that there exists at least one cyclic group whose order is not a power of a prime number which contains k but no more than k proper subgroups, where k is an arbitrary positive integer, is that $k+2$ is not a prime number.*

The simplest case which comes under this theorem is when $k=2$. In this case the cyclic group of order pq, p and q being distance prime numbers, contains 2 but no more than 2 proper subgroups. The only other group which contains exactly 2 proper subgroups is the cyclic group of order p, p being an arbitrary prime number. When $k=4$ the cyclic group of order pq^2 contains exactly 4 proper subgroups for every pair of values of p and q, but in this case the non-cyclic group of order 9 also contains exactly 4 proper subgroups as well as the cyclic group of order p^5. These three groups are the only abelian groups which separately contain exactly 4 proper subgroups. There are also two non-abelian groups which separately contain exactly 4 proper subgroups, viz., the quaternion group and the non-cyclic group of order 6. It is easy to prove that every non-abelian group contains at least as many as 4 proper subgroups, and that the only non-abelian groups which contain exactly 4 proper subgroups are the two mentioned in the preceding sentence.

In view of the rapid recent changes in mathematical interests it may be desirable to add here that in 1898 there appeared the first part of a large mathematical encyclopedia entitled *Encyklopädie der Mathematischen Wissenschaften* which was completed in 1935. Even before it was completed it was generally recognized that at least the first volume thereof was already out of date and in need of many fundamental modifications in order to be abreast with current scholarship along its line. Since the summer of 1939 several parts of a revised volume 1 of this encyclopedia have been published, and it is interesting to observe how profoundly these parts, relating to algebra and the theory of numbers, differ from the corresponding parts of the first edition. In this connection it may be noted that much more space is devoted to group theory and its applications in this revised edition than in the first edition.

While only about 18 pages are devoted to finite discrete groups in this first edition, 51 pages are devoted to only a part of this subject in the revised edition under the heading of general group theory. It is also significant that the subject of abstract groups precedes that of permutation groups in the revised edition while the reverse order was followed in the first edition. Notwithstanding the rapid changes in mathematical interests and the many superficial mathematical activities of recent years group theory seems to have established its fundamental usefulness in the minds of a large number of the most successful mathematical investigators of modern times. It is also important to note the large increase in the number of the different writers on this subject in recent years.

A GENERALISATION OF THE CIRCULAR AND HYPERBOLIC FUNCTIONS

BY

P. KESAVA MENON, *Department of Mathematics, University of Madras.*

1·1 The object of this paper is to study a class of functions which include the hyperbolic and circular functions as particular cases.

1·2 Let $P_r(x) = \frac{1}{n}\sum_{s=0}^{n-1} t^{-rs}\exp(m_s x)$ $(r = 0, 1, \cdots n-1)$, where $t = \exp(2\pi i/n)$ and $m_1, m_2 \cdots$ are unequal and different from zero.

The functions $P_0(x), \cdots P_{n-1}(x)$ are linearly independent solutions of an ordinary linear differential equation of order n with constant coefficients, since the n-rowed determinant, whose element in the $(r+1)$th row and the $(s+1)$th column is t^{-rs} is not equal to zero.

1·3 It is easily seen that

$$\sum_{r=0}^{n-1} t^{sr} P_r(x) = \exp(m_s x) \quad (s = 0, 1, 2, \cdots, n-1).$$

Multiplying together all these relations we get the circulantal identity

$$\begin{vmatrix} P_0, & P_1, & P_2, & \cdots\cdots, & P_{n-1} \\ P_{n-1}, & P_0, & P_1, & \cdots\cdots, & P_{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ P_1, & P_2, & \cdots & \cdots & P_0 \end{vmatrix} = \exp\left(\sum m_s x\right)$$

where P_r stands for $P_r(x)$. We note in particular that, when $\sum m_s = 0$, the circulant is identically equal to 1.

1·4 $P_r(x)$ has the addition formula given by

$$P_r(x+y) = P_r(x)\,P_0(y) + P_{r-1}(x)\,P_1(y) + \cdots + P_0(x)\,P_r(y) +$$
$$+ P_{n-1}(x)\,P_{r+1}(y) + P_{n-2}(x)\,P_{r+2}(y) + \cdots + P_{r+1}(x)\,P_{n-1}(y)$$

This may easily be demonstrated by replacing the P_s by their actual expressions.

1·5 It is seen from the identity

$$\left(P_0(x) + t\,P_1(x) + t^2 P_2(x) + \cdots + t^{n-1} P_{n-1}(x)\right) \times$$
$$\left(P_0(y) + t\,P_1(y) + \cdots + t^{n-1} P_{n-1}(y)\right)$$
$$= P_0(x+y) + t\,P_1(x+y) + \cdots + t^{n-1} P_{n-1}(x+y),$$

and the addition formulæ that, if the left side of the identity be multiplied out making use of the simplification $t^{n+\lambda} = t^{\lambda}$, then the terms which are coefficients of t^r on both sides are equal. It follows that the expression for $P_r(x + x_2 + \cdots + x_p)$ in terms of the $P_s(x_k)$ may be obtained by expanding the product $\prod\limits_{k=1}^{p} \sum\limits_{s=0}^{n-1} t^s P_s(x_k)$ making use of the simplification $t^{n+\lambda} = t^{\lambda}$ and collecting the terms which form the coefficient of t^r.

1·6 The geometrical significance of the determinantal identity is that the curve (in n-dimensional space) given by $x_r = A_r P_r(x)$ $(r = 0, 1, \cdots n-1)$, where the m_s in the P_s satisfy the condition $\sum\limits_{s=0}^{n-1} m_s = 0$, always lies on the algebraic manifold.

$$\begin{vmatrix} \dfrac{x_0}{A_0}, & \dfrac{x_1}{A_1}, & \cdots\cdots\cdots, & \dfrac{x_{n-1}}{A_{n-1}} \\ \dfrac{x_{n-1}}{A_{n-1}}, & \dfrac{x_0}{A_0}, & \cdots\cdots\cdots, & \dfrac{x_{n-2}}{A_{n-2}} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{x_1}{A_1}, & \dfrac{x_2}{A_2}, & \cdots\cdots\cdots & \dfrac{x_0}{A_0} \end{vmatrix} = 1$$

1·7 $P_r(x)$ is not in general periodic. If $m_1, m_2, \ldots, m_{n-1}$ are all integral multiples, (+ve or −ve) of m_0, $P_r(x)$ is evidently periodic with period $2\pi i/m_0$. In the general case, corresponding to the periodicity relation

$$f(x + 2\pi i/m_0) - f(x) = 0$$

of a function $f(x)$ with period $2\pi i/m_0$, we have a relation which may be stated as follows:

If $f(x)=A_0\, e^{m_0 x}+ A_1\, e^{m_1 x}+ \cdots + A_{n-1}\, e^{m_{n-1} x}$, then

$$f(x + \sum_{n} 2\pi i/m_r) - \sum_{n-1} f(x+ \sum 2\pi i/m_r) + \cdots + (-1)^n f(x) = 0,$$

where $\sum\limits_{t} 2\pi i/m_r$ denotes the sum of any t of the terms $2\pi i/m_r$ $(r=0,\ 1,\ \ldots\ n-1)$, and $\sum f\,(x+ \sum\limits_{t} 2\pi i/m_r)$ denotes the sum of all the ${}_nC_t$ functions $f(x+ \sum\limits_{t} 2\pi i/m_r)$ corresponding to a given t. In case some of the m_s are multiples of another m the above relation may be replaced by a simpler one.

To prove this result we note that if $f(x+2\pi i/m_0)-f(x)=f_1(x)$, say, then $f_1(x)$ is of the same form as $f(x)$ but does not contain the term $\exp(m_0 x)$ and similarly

$$f_1\,(x+2\pi i/m_1)-f_1\,(x)=f_2\,(x),\ \text{say},$$

$$f_{n-2}\,(x+2\pi i/m_{n-2})-f_{n-2}\,(x)=f_{n-1}\,(x),$$

where $f_{n-1}(x)$ is periodic with period $2\pi i/m_{n-1}$, and so

$$f_{n-1}\,(x+2\pi i\ m_{n-1})-f_{n-1}\,(x)=0.$$

Eliminating $f_1, f_2, \ldots, f_{n-1}$, we get the required result.

2·1. Let us consider the following particular cases of the P-functions. Let $m_r=\rho^r$, and then denote $P_r(x)$ by $H_r(x)$. We find that

$$H_r\,(x)= \frac{x^r}{r!} + \frac{x^{n+r}}{(n+r)!} + \frac{x^{2n+r}}{(n+2r)!} + \cdots,$$

which is real for real values of x. The differential equation for $H_r\,(x)$ is readily seen to be $\dfrac{d^n y}{dx^n} = y$.

Also $\qquad \dfrac{d}{dx} H_r\,(x) = H_{r-1}\,(x),\ (r \neq 0)$

and $\qquad \dfrac{d}{dx} H_0\,(x) = H_{n-1}\,(x).$

2·2. Let $m_r=e^{\frac{(2r+1)\pi i}{n}}$, and then let

$$P_r\,(x)=e^{r\pi i/n}\, C_r(x).$$

We have $C_r\,(x) = \dfrac{x^r}{r!} - \dfrac{x^{n+r}}{(n+r)!} + \dfrac{x^{2n+r}}{(2n+r)!} - \cdots,$

which is real for real values of x. The differential equation for $C_r\,(x)$ is

$$\frac{d^n y}{dx^n} + y = 0.$$

Also $$\frac{d}{dx} C_r(x) = C_{r-1}(x) \quad (r \neq 0)$$

and $$\frac{d}{dx} C_0(x) = -C_{n-1}(x).$$

The addition formula for $C_r(x)$ becomes

$$\begin{aligned} C_r(x+y) &= C_r(x)\,C_0(y) + C_{r-1}(x)\,C_1(y) + \cdots + C_0(x)\,C_r(y) \\ &\quad - C_{n-1}(x)\,C_{r+1}(y) - \cdots - C_{r+1}(x)\,C_{n-1}(y), \end{aligned}$$

and the circulantal identity takes the form

$$\begin{vmatrix} C_0, & C_1, & C_2, & \cdots\cdots\cdots & C_{n-1} \\ -C_{n-1}, & C_0, & C_1, & \cdots\cdots, & C_{n-2} \\ -C_{n-2}, & -C_{n-1}, & C_0, & \cdots\cdots, & C_{n-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -C_1, & -C_2, & -C_3, & \cdots\cdots & C_0 \end{vmatrix} = 1,$$

where all the terms below the leading diagonal have the negative sign and the rest have the positive sign.

3·1 It could be shown that $P_r(x)$ has an infinite number of zeroes if $n \geqslant 2$.

We proceed to show that $C_r(x)$ has at least one positive zero for $n > 1$. We do not consider the case $n=2$, since the C-functions are then merely the cosine and sine of x. Let n be greater than or equal to 3. Then

$$\frac{C_r(x)}{x^r} = \frac{1}{r\,!} - \frac{x^n}{(n+r)\,!} + \frac{x^{2n}}{(2n+r)\,!} - \cdots\cdots$$

which tends to $1/r\,!$ as $x \to 0$.

Consider the quadratic in x^n, viz.,

$$\frac{x^{2n}}{(2n+r)\,!} - \frac{x^n}{(n+r)\,!} + \frac{1}{r\,!} = 0.$$

The roots of this equation are given by

$$2\,(n+r)\,!\;\; x^n = (2n+r)\,! \;\pm \sqrt{\{(2n+r)\,!\}^2 - \frac{4\,(2n+r)\,!\,\{(n+r)\,!\}^2}{r\,!}}$$

Consider the root x_1^n, given by the negative sign. This is real and positive if

$$\frac{(2n+r)!}{(n+r)!} > 4\,\frac{(n+r)!}{r!} \qquad (r=0,\ 1,\ \cdots\ n-1)$$

i.e. if $$\left(1+\frac{n}{r+1}\right)\left(1+\frac{n}{r+2}\right)\cdots\left(1+\frac{n}{r+n}\right) > 4.$$

This inequality will be satisfied for $r=0,\ 1,\cdots n-1$ if it is satisfied for $r=n-1$. Now

$$\left(1+\frac{n}{n}\right)\left(1+\frac{n}{n+1}\right)\cdots\left(1+\frac{n}{2n-1}\right)$$
$$= 4\cdot\frac{(2n+1)}{(n+2)}\cdot\frac{(2n+3)\,(2n+4)\cdots(3n-1)}{(n+3)\,(n+4)\cdots\cdots(2n-1)} > 4.$$

Thus the quadratic in x^n has a positive root given by x_1^n. Now

$$\frac{C_r(x)}{x^r} = \frac{1}{r} - \frac{x^n}{(n+r)!} + \frac{x^{2n}}{(2n+r)!} - x^{3n}\left\{\frac{1}{(3n+r)!} - \frac{x^n}{(4n+r)!}\right\}$$
$$- x^{5n}\left\{\qquad\right\} - \cdots\cdot$$

The expression within each of the flower brackets is positive for $x^n=(2n+r)!\,/\,2(n+r)!$ and so for $x=x_1$. Hence $C_r(x)/x^r$ is negative for $x=x_1$. Thus there is a root of $C_r(x)$ between 0 and x_1

Since $$\frac{C_r(x)}{x^r} = \frac{1}{r!} - \frac{x^n}{(r+n)!} + x^{2n}\left\{\frac{1}{(2n+r)!} - \frac{x^n}{(3n+r)!}\right\}$$
$$+ x^{4n}\left\{\qquad\right\} + \cdots\cdot,$$

and the expression within each of the flower brackets is negative for $x^n = \frac{(n+r)!}{r!}$, it follows that $\frac{C_r(x)}{x^r}$ is positive for $x^n = \frac{(n+r)!}{r!}$. Therefore there is a root of $C_r(x)$ between $\left\{\frac{(n+r)!}{r!}\right\}^{1/n}$ and x_1.

3·2 It is quite probable that there are an infinite number of real roots for $C_r(x)$. We can readily show that, when $n=4$,

$$C_0(x) = \prod_{n=1}^{\infty}\left\{1-\frac{4x^4}{(2n-1)^4\,\pi^4}\right\}$$

and $$C_2(x) = \frac{x^2}{2}\prod_{n=1}^{\infty}\left\{1-\frac{x^4}{4n^4\,\pi^4}\right\}.$$

It follows from this and the fact that $\frac{d}{dx} C_2(x) = C_1(x)$ and $\frac{d}{dx} C_0(x) = -C_3(x)$ that all the four functions $C_r(x)$ $(r = 0, 1, 2, 3)$ have an infinite number of real roots.

4. The following particular cases and examples may be found interesting.

Let n be equal to 3. Then

$$H_0^3(x) + H_1^3(x) + H_2^3(x) - 3\,H_0(x)\,H_1(x)\,H_2(x) = 1.$$
$$H_0(2x) = H_0^2(x) + 2H_1(x)\,H_2(x)$$
$$H_0(3x) = 1 + 9\,H_0(x)\,H_1(x)\,H_2(x).$$
$$H_0(-x) = H_0^2(x) - H_1(x)\,H_2(x).$$

If $H_0(\lambda) = H_1(\mu) = H_2(\nu) = 0$, and $H_r(\lambda)$ is denoted by λ_r etc., then

$$H_0(x + \lambda) = \lambda_1 H_2(x) + \lambda_2 H_1(x),$$
$$H_0(3\lambda) = H_0(3\mu) = H_0(3\nu) = 1,$$
$$H_0(\lambda + \mu + \nu) = \lambda_1\,\mu_2\,\nu_0 + \lambda_2\,\mu_0\,\nu_1.$$

The periodicity law reduces to

$$H_r(x + 2\pi i) + H_r(x + 2\pi i\omega) + H_r(x + 2\pi i\omega^2)$$
$$= H_r(x - 2\pi i) + H_r(x - 2\pi i\omega) + H_r(x - 2\pi i\omega^2),$$

where 1, ω, ω^2 are the cube roots of unity.

I thank Dr. R. Vaidyanathaswamy for his valuable guidance in the conduct of this investigation.

ON ABEL'S DIVERGENCE TEST FOR SERIES OF POSITIVE TERMS

BY

C. T. RAJAGOPAL, *Madras Christian College, Tambaram.*

1. Of the three theorems proved here, the first is a simple generalization of Abel's "test":

If $a_n \downarrow 0$ *as* $n \to \infty$, *the condition* $\lim n\, a_n = 0$ *is necessary for the convergence of* $\sum^{\infty} a_n$.

It is contained in the integral analogue of the test:

THEOREM A.—*If* $f(x) \downarrow 0$ *as* $x \to \infty$, *the condition* $\lim x\, f(x) = 0$ *is necessary for the convergence of* $\int^{\infty} f(x)\, dx$.

The second theorem is a generalization of certain results of Hardy and the last a deduction from one of Kronecker.

1·1. In what follows, the sequences $\{D_n\}$, $\{d_n\}$ are defined:

$$0 = D_0 < D_1 < D_2 < \cdots < D_n \to \infty,$$
$$d_n \equiv D_n - D_{n-1} \ (n \geqslant 1).$$

The starting point of our inquiry is

2. LEMMA 1 (KRONECKER). *The condition*

$$\lim\, (a_1 D_1 + a_2 D_2 + \cdots + a_n D_n)/D_n = 0$$

is necessary for the convergence of $\sum^{\infty} a_n$ (a_n *real or complex*). [1, H 10.]

2·1. If $a_n \downarrow 0$, we can take $D_n = \dfrac{1}{a_n}$ in the lemma and deduce therefrom Abel's test. More generally, taking

$$D_n = \frac{1}{(n \cdot l_1 n\, l_2 n \cdots l_k n)\, a_n},$$

where $l_1 n = \log n$, $l_2 n = \log\log n$ etc. and $n \geqslant m$ such that $l_{k+1} m > 0$, we infer:

If $(n \cdot l_1 n \cdot l_2 n \cdot \cdot \cdot \cdot l_k n)\, a_n \downarrow 0$, *the monotony asserting itself from a certain stage on, the condition* $\lim\, (n \cdot l_1 n \cdot l_2 n \cdot \cdot \cdot \cdot l_{k+1} n) a_n = 0$ *is necessary for the convergence of* $\sum\limits^{\infty} a_n$. [Cp. 1, B 18]

2·2. The conclusion of the lemma does not imply $\lim \dfrac{a_n D_n}{d_n} = 0$ although $\lim \dfrac{a_n D_n}{d_n} = 0$ implies this conclusion. Even if we restrict ourselves to positive series, the condition $\lim \dfrac{a_n D_n}{d_n} = 0$ is not necessary for the convergence of $\sum a_n$, although the condition $\underline{\lim} \dfrac{a_n D_n}{d_n} = 0$ is. It is not difficult to construct examples of convergent series $\sum a_n$ $(a_n > 0)$ for which there is a $\{D_n\}$ such that $\lim \dfrac{a_n D_n}{d_n} = \infty$. Consider, for instance, the convergent series given by

$$a_n \begin{cases} = \dfrac{1}{\log n} & \text{when } n \text{ is of the form } 2^{m^2}, \\ = \dfrac{1}{n^2} & \text{otherwise.} \end{cases}$$

Taking $d_n = n^{-k}$ $(0 < k < 1)$, we find $D_n \sim \dfrac{n^{1-k}}{1-k}$ so that

$$\frac{a_n D_n}{d_n} \sim \frac{1}{1-k} \cdot \frac{n}{\log n} \rightarrow \infty \text{ as } n = 2^{m^2} \rightarrow \infty.$$

Theorems 1 and 3 suggest a relation between a_n (> 0) and d_n sufficient to ensure that the convergence of $\sum a_n$ shall imply $\lim \dfrac{a_n D_n}{d_n} = 0$.

3. **THEOREM 1.** *If* $\dfrac{a_n}{d_n} \downarrow 0$, *the condition* $\lim \dfrac{a_n D_n}{d_n} = 0$ *is necessary for the convergence of* $\sum\limits^{\infty} a_n$.

Proof.—This may be deduced from Theorem A by taking

$$f(x) = \frac{a_n}{d_n} \text{ for } D_{n-1} < x < D_n.$$

COROLLARY 1. $\sum a_n \equiv \sum \dfrac{d_n}{D_n}$ *is divergent* (since $\dfrac{a_n D_n}{d_n} = 1$)

COROLLARY 2. The results in §2·1 can be deduced from Theorem 1 also by taking $d_n = 1, \dfrac{1}{n}, \dfrac{1}{n \cdot l_1 n}, \cdot \cdot \cdot \cdot \cdot \cdot$ successively.

3·1. It is easy to prove that, if $d_n = O(1)$, the condition

$$\left.\begin{aligned} &\frac{1}{d_n}\log\frac{a_{n+1}/d_{n+1}}{a_n\, d_n} \\ \text{or } &\frac{1}{d_n}\left(\frac{a_{n+1}\, d_{n+1}}{a_n/d_n} - 1\right)\end{aligned}\right\} = -\frac{\mu}{D_n} - O\left(\frac{1}{D_n^{\lambda}}\right) \qquad (\lambda > 1),$$

which is much narrower than the monotony of $\left\{\frac{a_n}{d_n}\right\}$, makes $\lim \frac{a_n D_n}{d_n} = 0$ *necessary and sufficient* for the convergence of $\sum a_n$ [3, 461-462.]

The last remark applies, for instance, to series such as

$$\sum a_{n+1} \equiv \sum d_{n+1} \frac{\prod_{\nu=1}^{n}(\alpha d_\nu + D_\nu)(\beta d_\nu + D_\nu)}{\prod_{\nu=1}^{n}(\gamma d_\nu + D_\nu)(\delta d_\nu + D_\nu)} \quad \left(\alpha,\ \beta,\ \gamma,\ \delta \text{ real}; \ \gamma_1\delta \neq -\frac{D_1}{d_1},\ \frac{-D_2}{d_2},\ \ldots\right)$$

which reduce to the Hypergeometric type when $d_n = 1$.

4. THEOREM 2 (*a*). *If* $\frac{a_n}{d_n} \downarrow 0$, $\sum^{\infty} a_n$ *is convergent, then* $\sum_{1}^{\infty} D_n\left(\frac{a_n}{d_n} - \frac{a_{n+1}}{d_{n+1}}\right)$ *converges to the same sum as* $\sum_{1}^{\infty} a_n$.

(*b*). *If, in addition,* $\left\{\frac{1}{d_n}\left(\frac{a_n}{d_n} - \frac{a_{n+1}}{d_{n+1}}\right)\right\}$ *is steadily decreasing, then* $\frac{a_n}{d_n} - \frac{a_{n+1}}{d_{n+1}} = o\left(\frac{d_n}{D_n^2}\right)$.

Proof. (*a*) Setting $b_n = D_n\left(\frac{a_n}{d_n} - \frac{a_{n+1}}{d_{n+1}}\right) > 0$, we find

$$b_n = \frac{a_n D_n}{d_n} - \frac{a_{n+1} D_{n+1}}{d_{n+1}} + a_{n+1}$$

and
$$\sum_{\nu=1}^{n} b_\nu = \sum_{\nu=1}^{n+1} a_\nu - \frac{a_{n+1} D_{n+1}}{d_{n+1}},$$

whence the first half of the theorem follows......

(*b*) The given condition

$$\frac{1}{d_n}\left(\frac{a_n}{d_n} - \frac{a_{n+1}}{d_{n+1}}\right) \geqslant \frac{1}{d_{n+1}}\left(\frac{a_{n+1}}{d_{n+1}} - \frac{a_{n+2}}{d_{n+2}}\right)$$

can be proved equivalent to

$$\frac{b_n}{d_n} + \frac{a_{n+1}}{d_{n+1}} \geqslant \frac{b_{n+1}}{d_{n+1}} + \frac{a_{n+2}}{d_{n+2}}$$

Now the convergence of $\sum a_n$ implies that of $\sum \frac{a_{n+1}}{d_{n+1}} d_n$ since $\frac{a_{n-1}}{d_{n+1}} d_n \leqslant a_n$.

Hence $\sum a'_n \equiv \sum \left(b_n + \frac{a_{n+1}}{d_{n+1}} d_n \right)$ is convergent and $\left\{ \frac{a'_n}{d_n} \right\}$ is steadily decreasing.

Therefore, by Theorem 1,

$$\frac{b_n D_n}{d_n} < \frac{a'_n D_n}{d_n} \to 0.$$

COROLLARY 1. The case $d_n = 1$ [1, B 20] is due to Hardy.

With $d_n = 1/n \cdot l_1 n \cdot l_2 n \cdot \cdot \cdot l_k n$, we obtain

COROLLARY 2 (*a*). *If* $\sum\limits^{\infty} a_n$ *is convergent and* $(n \cdot l_1 n \cdot l_2 n \cdot \cdot \cdot \cdot l_k n)\, a_n \downarrow 0$, *then*

$$\sum^{\infty} l_{k+1} n \, [n \cdot l_1 n \cdot \cdot \cdot l_k n \cdot a_n - (n+1) \cdot l_1 (n+1) \cdot \cdot \cdot l_k (n+1) \cdot a_{n+1}]$$

is also convergent.

(*b*) *If, in addition, the sequence*

$$\{n \cdot l_1 n \cdot \cdot \cdot l_k n \, [n \cdot l_1 n \cdot \cdot \cdot l_k n \cdot a_n - (n+1) \cdot l_1 (n+1) \cdot \cdot \cdot l_k (n+1)\, a_{n+1}]\}$$

is steadily decreasing, then

$$n \cdot l_1 n \cdot \cdot \cdot l_k n \cdot a_n - (n+1) \cdot l_1 (n+1) \cdot \cdot \cdot l_k (n+1) \cdot a_{n+1} = o\,(1/n \cdot l_1 n \cdot \cdot \cdot l_k n \cdot l^2_{k+1} n)$$

4·1. Bromwich (*Infinite Series*, ed. 2, p. 430 ex. 5) observes that if $a_n > 0$ and $\left\{ \frac{a_n}{d_n} \right\}$ steadily decreases, the divergence of $\sum b_n \equiv \sum D_n \left(\frac{a_n}{d_n} - \frac{a_{n+1}}{d_{n+1}} \right)$ is secured by that of $\sum a_n$. If $\lim \frac{a_n}{d_n} = 0$, it is not difficult to prove this statement. If, however, $\lim \frac{a_n}{d_n} = \rho > 0$, the statement is not necessarily true as can be seen by taking $a_n = \frac{\rho}{n} + \frac{1}{n \cdot l_1^2 n}$, $d_n = \frac{1}{n}$, in which case $\sum b_n$ is convergent. Hence Theorem 2 (*a*) can be completed by

THEOREM 2 (*a'*). *If* $\frac{a_n}{d_n} \downarrow 0$, *then* $\sum\limits^{\infty} a_n$ *and* $\sum\limits^{\infty} D_n \left(\frac{a_n}{d_n} - \frac{a_{n+1}}{d_{n+1}} \right)$ *converge or diverge together.*[*]

---

[*] The divergence part of Theorem 2 (*a'*) can be used to prove an important limit theorem of Cesàro (Bromwich, *loc. cit.*)

Corollary 2 (*a*) can be similarly completed.

5. The theorem in this section is established by a lemma which is an immediate deduction from a theorem of Minakshisundaram [2, Theorem 1 (ii)].

5·1. LEMMA 2. *If (i)* $\{\lambda_n\}$, $\{\mu_n\}$ *are the sequences:*

$$0=\lambda_0 < \lambda_1 < \cdots < \lambda_n \to \infty,$$

$$\mu_n=\lambda_n-\lambda_{n-1} \ (n>1);$$

(ii) $\{S_n\}$ *is a real sequence such that* $S_{n-1}-S_n < O\left(\frac{\mu_n}{\lambda_n}\right)$;

(iii) $(\mu_2 S_1+\mu_3 S_2+ \cdots +\mu_n S_{n-1})/\lambda_n \to 0$;

then $\overline{\lim}\ S_n=0$

The reader will have no difficulty in supplying the proof.

5·2. THEOREM 3. *If* $\sum^{\infty} a_n \ (a_n>0)$ *is convergent,*

$$\frac{a_n}{d_n}-\frac{a_{n+1}}{d_{n+1}} < O\left(\frac{d_n}{D_n^2}\right), \text{ then } \lim \frac{a_n D_n}{d_n}=0.$$

Proof. The given condition

$$\frac{a_n}{d_n}-\frac{a_{n+1}}{d_{n+1}} < K\frac{d_n}{D_n^2} \qquad (0<K<\infty)$$

makes

$$\frac{a_n D_n}{d_n}-\frac{a_{n+1} D_{n+1}}{d_{n+1}} < \frac{a_n D_n}{d_n}-\frac{a_{n+1} D_n}{d_{n+1}}$$

$$< K\frac{d_n}{D_n}.$$

Hence if we replace λ_n, μ_n, S_n in Lemma 2 by D_n, d_n, $\frac{a_{n+1} D_{n+1}}{d_{n+1}}$ respectively, (ii) is satisfied.

In consequence of Kronecker's theorem, (iii) is also satisfied.

Lemma 2 therefore establishes that $\lim \frac{a_n D_n}{d_n}=0=\underline{\lim}\frac{a_n D_n}{d_n}$, which is the result sought.

Remark. Theorem 3 gives us some idea of the gap between a condition (N) which makes $\lim \frac{a_n D_n}{d_n}=0$ *necessary* for the convergence of $\sum^{\infty} a_n$ and a condition (N − S) which makes $\lim \frac{a_n D_n}{d_n}=0$

necessary and sufficient for the convergence of $\sum a_n$. When $d_n = O(1)$, (N – S) in the form

$$\log \frac{a_{n+1}/d_{n+1}}{a_n/d_n} = -\mu \frac{d_n}{D_n} - O\left(\frac{d_n}{D_n^{\lambda}}\right) \qquad (\lambda > 1)$$

can be shown to involve

$$\frac{a_n}{d_n} - \frac{a_{n+1}}{d_{n+1}} < O\left(\frac{d_n}{D_n^{\mu+1}}\right) \qquad (\mu > 1).$$

On the other hand, the (N) in Theorem 3 is

$$\frac{a_n}{d_n} - \frac{a_{n+1}}{d_{n+1}} < O\left(\frac{d_n}{D_n^{2}}\right)$$

6. In conclusion, I state integral analogues of some of the results proved above, leaving their proofs to the reader.

THEOREM B. *The condition* $\lim_{x\to\infty} \frac{\int^x t f(t)\,dt}{x} = 0$ *is necessary for the convergence of* $\int^\infty f(x)\,dx$.

THEOREM C (*a*). *If* $f(x) \downarrow 0$, *then* $\int^\infty f(x)\,dx$ *and* $\int^\infty -x f'(x)\,dx$ *converge or diverge together.*

(*b*) *If* $f(x) \downarrow 0$, $\int^\infty f(x)\,dx$ *converges*, $f''(x) \geq 0$, *then*

$$-f'(x) = o(1/x^2).$$

THEOREM D. *If* $\int^\infty f(x)\,dx$ $(f > 0)$ *is convergent*,

$$-f'(x) < O(1/x^2), \text{ then } \lim x f(x) = 0.$$

References

[1] FRANCIS, E. C. and LITTLEWOOD, J. E. *Examples in infinite series with solutions* (Cambridge, 1928).

[2] MINAKSHISUNDARAM, S. A Tauberian theorem on (λ, k)—process of summation, *Journ. Indian Math. Soc.* (*New Series*), 3 (1938), 127-130.

[3] RAJAGOPAL, C. T. Some theorems connected with Maclaurin's integral test, *Math. Gazette*, 23 (1939), 456-461.

COMPATIBILITY OF FISHER'S TESTS FOR INDEX NUMBER FORMULAE.*

BY

S. SUBRAMANIAN

Introduction

The problem of averaging the fluctuations in the prices or quantities (produced or consumed) of several commodities has been engaging the attention of statistical workers since long ago and various formulae have been put forward from time to time as suitable for the purpose. The unweighted arithmetic mean held the first place among these till recently with the simple geometric mean as a close second. These have now given place to the weighted arithmetic and geometric means. Further improvements have been suggested by Professor Irving Fisher in the form of the principle of 'crossing' and the application of the several tests for the soundness of a formula. These tests are seven in number and are called**

(1) the identity test;
(2) time reversal test;
(3) base test;
(4) circular test;
(5) commensurability test;
(6) determinateness test;
(7) factor reversal test.

If P_{ts} denotes the price index for year t with respect to the base year s and is given by $f\,(p_{1t},\ p_{1s},\ p_{2t},\ p_{2s},\ldots\ldots q_{1t},\ q_{1s},\ \ldots\ldots)$ where the p's are the prices and the q's, the quantities, then test (1) requires that $P_{tt}=1$; test (2), $P_{ts}.\ P_{st}=1$; test (3) $\frac{\partial}{\partial s}\left(\frac{P_{ts}}{P_{t's}}\right)=0$; test (4), $P_{ts}/P_{t's}=P_{tt'}$; test (5), $P_{ts}=f\,(\lambda_1\ p_{1t},\ \lambda_1\ p_{1s},\ \frac{q_{1t}}{\lambda_1},\ \frac{q_{1s}}{\lambda_1},\ldots\ldots\ldots\ldots)$; (6) P_{ts} should not become zero or infinite or indefinite when some price or quantity becomes zero; and (7), if p and q are interchanged in P_{ts} we should get Q_{ts} where $P_{ts}.\ Q_{ts}=\sum p_t\ q_t \div \sum p_s\ q_s$

---

* Paper read at the Conference of the Indian Mathematical Society (Lucknow, March 1938).

** R. Frisch: Journal of the American Statistical Association (1930).

It can be seen at once that if test (4) is satisfied, tests (1), (2) and (3) are automatically satisfied. Thus conditions (4), (5), (6) and (7) are the only independent ones. There is a widespread belief that some of these tests are incompatible with one another. Professor Frisch has observed[*] that tests (3), (5) and (6) cannot hold good at the same time. But as I have pointed out elsewhere[†] his conclusion is rendered incorrect by the fact that the solution of his differential equations is subject to certain limitations. The object of the present note is to show that the tests are perfectly compatible with one another and that a formula can be found so as to satisfy all of them. In part I of the note it is shown that a formula satisfying tests (4), (5) and (7) can also satisfy test (6); in part II some general remarks are offered concerning such a formula and its relation with Prof. Irving Fisher's ideal index.

Part I

1. The general form of P_{ts} satisfying tests (3) and (5) when no p or q vanishes, is given by

$$F\,(a_{1t},\ a_{2t},\ldots\ldots) \times G\,(a_{1s},\ a_{2s},\ldots)\cdot\cdot \left(\frac{p_{1t}}{p_{1s}}\right)^{c_1}\cdot\left(\frac{p_{2t}}{p_{2s}}\right)^{c_2}\ldots\ \ldots\ \ldots$$

where the a's are the values and F, G are arbitrary functions.[‡]

When, for instance, some p_t vanishes, the corresponding a is omitted from F and the corresponding price ratio is replaced by the inverse quantity ratio[§].

Since test (4) is satisfied if the tests (3) and (1) are simultaneously satisfied, by applying test (1) to the form given above we get a formula satisfying tests (4) and (5). P_{ts} now takes the form

$$\frac{F\,(a_{1t},\ a_{2t},\ \ldots\ .)}{F\,(a_{1s},\ a_{2s},\ \ldots\ .)}\cdot\left(\frac{p_{1t}}{p_{1s}}\right)^{c_1}\cdot\left(\frac{p_{2t}}{p_{2s}}\right)^{c_2}\ \ldots\ .$$

when no p or q vanishes. When some p_t is zero the corresponding a_t and a_s are to be omitted and the corresponding price ratio is to be replaced by the inverse quantity ratio.

2. If we now apply test (7) we get

$$P_{ts} = \left(\frac{p_{1t}}{p_{1s}}\right)^{c_1}\cdot\left(\frac{p_{2t}}{p_{2s}}\right)^{c_2}\ \ldots\ \times \sqrt{\frac{(\sum p_t\ q_t) \div \sum p_s\ q_s}{\left(\frac{a_{1t}}{a_{1s}}\right)^{c_1}\left(\frac{a_{2t}}{a_{2s}}\right)^{c_2}}}\ \ldots\ .\ \text{(A)}$$

---

[*] Frisch: *op. cit.*, p. 404.

[†] Subramanian: *Journal of the American Statistical Association* (1934).

[‡] Frisch: *op. cit.*, p. 404.

[§] Subramanian: *op. cit.*

It can be seen that if some p_t is zero, say p_{rt},

$$P_{ts} = \left(\frac{p_{1t}}{p_{1s}}\right)^{c_1} \cdot \left(\frac{p_{2t}}{p_{2s}}\right)^{c_2} \cdots \left(\frac{q_{\cdot s}}{q_{rt}}\right)^{c_r} \cdots \times \sqrt{\frac{\left(\sum p_t\ q_t\right) \div \sum p_s\ q_s}{\left(\frac{a_{1t}}{a_{1s}}\right)^{c_1} \cdots \left(\frac{a_{rs}}{a_{rt}}\right)^{c_r}}} \quad \cdots\cdots \text{(B)}$$

It must be observed that all the c's are perfectly arbitrary constants in both (A) and (B). These constants can be so adjusted that P_{ts} takes a value other than zero even when some p or q vanishes. For instance in form (A) when p_{rt} is made zero the only factors that are affected are $(p_{rt}/p_{rs})^{c_r}$ in the numerator and $(a_{rt}\ a_{rs})^{\frac{1}{2}c_r}$ in the denominator. These will give the result $k\,(p_{rt})^{\frac{1}{2}c_r}$* where k has a value other than zero. Now if we make c_r also zero, as we may, we see that P_{ts} does not vanish. Hence form (A) satisfies test (6). Form (B) can similarly be shown to satisfy test (6).

Thus forms (A) and (B) satisfy all the tests suggested by Professor Irving Fisher. It may be mentioned in this connection that it is unusual to provide, in a price index, for a commodity whose price is zero at any time, Therefore the determinateness test is not of much practical interest However, it is interesting to find that even when such a commodity is included, the index number can take a value other than zero. The surprising fact about forms (A) and (B) is that they are in geometric form and still satisfy the determinateness test.

Part II

A word may now be said about Professor Fisher's ideal index number. Since this index is essentially arithmetic in form it will not satisfy the circular test. The index takes the form $\sqrt{\frac{\sum p_t\ q_t}{\sum p_s\ q^s} \times \frac{\sum p_t\ q_s}{\sum p^s\ q^t}}$. It can be easily verified that form (A) in the present note can be written also as $\sqrt{\frac{\sum p_t\ q_t}{\sum p_s\ q_s} \cdot \frac{\pi\ (p_t\ q_s)^c}{\pi\ (p_s\ q_t)^c}}$. The essential difference between the two forms mentioned above is that in the first, the second factor is of an aggregative type which can also be considered as arithmetic whereas in the second form, the corresponding factor is of a geometric type. If the determinateness test is excluded, form (B) is rendered unnecessary and form (A) would do. As the latter now satisfies all the tests, it can very well be called the ideal index number. Of course, it is necessary to reach some agreed solution beforehand with regard to the arbitrary constants in the formula.

---

* The practice nowadays is to weight each price ratio with the corresponding value. If $\frac{p_{rt}}{p_{rs}}$ were weighted with $p_{rt}\ q_{rt}$ the question reduces to finding the limit of x^x as $x \to 0$. We know $\underset{x\to 0}{\mathrm{Lt}} \log x^x = \underset{x\to 0}{\mathrm{Lt}}\ x \log x = \underset{x\to 0}{\mathrm{Lt}} \frac{\log x}{1/x} = \underset{x\to 0}{\mathrm{Lt}} - \frac{1}{x} \Big/ \frac{1}{x^2}$

$= \underset{x\to 0}{\mathrm{Lt}}$ of $x = 0$; thus $x^x \to 1$ as $x \to 0$.

If some p_s is zero, then $p_s\ q_s$ will be the proper weight.

Conclusion

We have seen (i) that it is quite possible to find a formula satisfying all of Professor Irving Fisher's tests and that it is in the main of the *weighted* geometric type and (ii) that failure to pay proper attention to the weights has been responsible for the belief that some of Professor Fisher's tests are mutually incompatible.

ON THE FUNCTION DEFINED BY THE RELATION

$$\phi_r(x) = x\phi'_{r-1}(x)$$

BY

H. SREENIVASA SASTRY

1. The object of this paper is to find a general formula for the function $\phi_r(x)$ defined by the relation

$$\phi_r(x) = x\,\phi'_{r-1}(x), \qquad \dots \quad (1)$$

in terms of $\phi_0(x)$ and its successive derivatives. We obtain by differentiation

$$\left.\begin{aligned} &\phi_1(x) = x\,\phi'_0(x) \\ &\phi_2(x) = x\,\phi'_0(x) + x^2\,\phi_0''(x) \\ &\dots\dots\dots\dots\dots\dots \quad \dots\dots\dots\dots\dots\dots \\ &\phi_r(x) = a_{r,1}\,x\phi_0'(x) + a_{r,2}\,x^2\phi_0''(x) + \cdots + a_{r,r}\,x^r\phi_0^{(r)}(x) \end{aligned}\right\} \dots \quad (2)$$

where $a_{r,1}, a_{r,2}, \cdots, a_{r,r}$ are non-negative integers independent of x and $\phi_0(x)$ with

$$a_{r,1} = a_{r,r} = 1. \qquad \dots \quad (3)$$

11. To find a general formula for $a_{r,k}$, we note that $a_{r,k}$ is independent of $\phi_0(x)$, and hence we shall choose $\phi_0(x) = e^x$. From (2) we get

$$\sum_{n=1}^{\infty} \frac{n^r\,x^n}{n!} = \sum_{k=1}^{n} a_{r,k}\,x^k\,\phi_0^{(k)}(x) = \left(\sum_{k=1}^{r} a_{r,k}\,x^k\right)e^x,$$

which gives

$$\sum_{k=1}^{r} a_{r,k}\, x^k \equiv e^{-x} \sum_{n=1}^{\infty} \frac{n^r x^n}{n!}. \qquad \dots \quad (4)$$

Equating coefficients of x^k $(1 \leqslant k \leqslant r)$ in (4) we get the relation

$$a_{r,k} = \frac{k^r}{k!} - \frac{(k-1)^r}{(k-1)!} + \frac{(k-2)^r}{(k-2)!\,2!} - \cdots + \frac{(-1)^{k-1}\cdot 1^r}{(k-1)!\,1!}$$

$$= \frac{\Delta^k\, k^r}{k!}. \qquad \dots \quad (4')$$

where $\Delta f(k) = f(k) - f(k-1)$ and $\Delta^n = \Delta \cdot \Delta^{n-1}$. The above relation can also be written as*

$$a_{r,k} = \frac{k^{r-1}}{(k-1)!} - \frac{(k-1)^{r-1}}{(k-2)!} + \frac{(k-2)^{r-1}}{(k-2)!\,2!} - \cdots$$

$$= \frac{\Delta^{k-1}\, k^{r-1}}{(k-1)!}. \qquad \dots \quad (4'')$$

2. By comparing $\phi_r(x)$ and $x\,\phi'_{r-1}(x)$ in (2) we get the recurrence relation

$$a_{r,k} = a_{r-1,k-1} + k\, a_{r-1,k}. \qquad \dots \quad (5)$$

From which we deduce that

$$a_{r,k} = a_{r-1,k-1} + k\, a_{r-2,k-1} + \cdots + k^{r-k-1} a_{k,k-1} + k^{r-k} a_{k,k}. \qquad \dots \quad (6)$$

By using (3), (5) and (6) we can prove by double induction that

$$a_{r,k} = k^{r-k} + (k-1)\,\Delta k^{r-k} + \frac{k-2}{2!}\,\Delta^2 k^{r-k+1} + \cdots + \frac{\Delta^{k-1}\, k^{r-2}}{(k-1)!} \qquad \dots \quad (7)$$

which gives another formula for $a_{r,k}$. The relation (7) can also be verified more easily by collecting the powers of $(k-p)$ which occur explicitly in (7) and showing that it is actually equal to that in (4).

2·1. Taking $k = r$ in (3′) and using (3) we get the well-known result

$$r^r - r\,(r-1)^r + \frac{r\,(r-1)}{2!}\,(r-2)^r - \cdots = r! \qquad \dots \quad (8)$$

2·2. Taking $k = r-1$ in (7) we obtain

$$a_{r,r-1} = (r-1) + (r-2) + (r-3) + \cdots + 1$$

$$= \frac{r\,(r-1)}{2}. \; . \; . \; . \qquad \dots \quad (9)$$

---

* I am indebted to Dr. C. N. Srinivasa Iyengar for the elegant relation (4) and for help in this investigation.

which, in conjunction with (4) for $a_{r+1,r}$ gives

$$r^r - (r-1)(r-1)^r + \frac{(r-1)(r-2)}{2!}(r-2)^r - \cdots = (r+1)! \quad \ldots \quad (10)$$

2·3. From (5) we get successively

$$\begin{aligned} a_{r+2,r} &= a_{r+1,r-1} + r\ a_{r+1,r} \\ &= a_{r,r-2} + (r-1)\ a_{r,r-1} + r\ a_{r+1,r} \\ &\quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ &= a_{2,1} + 2a_{3,2} + 3a_{4,3} + \cdots + r\ a_{r+1,r} \end{aligned}$$

From which using (3) and (9) we get

$$a_{r+2,r} = \sum_{n=1}^{r} \frac{1}{2} n^2 (n+1) = \frac{1}{24} r (r+1)(r+2)(3r+1).$$

But by (4)

$$a_{r+2,r} = \frac{\Delta^{r-1} r^{r+1}}{(r-1)!}$$

so that we get

$$\begin{aligned} r^{r+1} - (r-1)(r-1)^{r+1} + \frac{(r-1)(r-2)}{2!}(r-2)^{r+1} - \cdots \\ = \frac{1}{24}(3r+1)(r+2)!. \quad \ldots \quad (11) \end{aligned}$$

We can get similar relations by considering $a_{r+3,r}$, etc. We may regard (10) and (11) as extensions of the known result (8).

3. THEOREM. *The integers $a_{r,2}, a_{r,3}, \cdots a_{r,r-1}$ are all divisible by r when it is a prime number.*

For we have, for $2<k<r-1$,

$$a_{r,k} = \frac{1}{(k-1)!}\left\{k^{r-1} - (k-1)(k-1)^{r-1} + \frac{(k-1)(k-2)}{2!}(k-2)^{r-1} - \cdots + (-1)^{k-1}\right\},$$

while

$$\begin{aligned} 0 &= (1-1)^{k-1} \\ &= 1 - (k-1) + \frac{(k-1)(k-2)}{2!} - \cdots + (-1)^{k-1} \quad \ldots \quad (12) \end{aligned}$$

Multiplying (12) by $\frac{1}{(k-1)!}$ and subtracting from the expression for $a_{r,k}$ we get

$$a_{r,k} = \frac{1}{(k-1)!}\left\{(k^{r-1} - 1) - (k-1)[(k-1)^{r-1} - 1] + \cdots + (-1)^{k-2}[2^{r-1} - 1]\right\}$$

From which we deduce that $a_{r,k}$ is divisible by r since by Fermat's theorem $p^{r-1}-1$ is divisible by r when r is a prime number and $p=2, 3, \ldots, r-1$.

3·1. Taking $\phi_0(x)=1+x+x^2+\cdots+x^n$ we get the formula $\phi_r(1)=1^r+2^r+\cdots+n^r$

$$=\sum_{k=1}^{r}\frac{1}{(k-1)!}\Delta^{k-1}k^{r-1}\phi_0^{(k)}(1),$$

where

$$\phi_0^{(k)}(1)=1\cdot2\cdots k+2\cdot3\cdot(k+1)+\cdots+(n-k+1)(n-k+2)\cdots n\cdot$$

$$=\frac{(n-k+1)(n-k+2)\cdots n\cdot(n+1)}{k+1}.$$

GLEANING

I hope I shall not create too many enemies if I flatly say that in mathematics classes, more often than in others, pupils are thought stupid by their teachers. In algebra and geometry one has to see relationships and act upon them quickly. But these are relationships of a special kind. A brilliant wit owes much to the ability to see unusual relationships. Some mathematicians have been brilliant wits. But not all.

T. H. PEAR: *The Art of Study.* p. 101.

(From A. A. KRISHNASWAMI IYENGAR, *Mysore*)

NOTES AND DISCUSSIONS

A Problem in Combinations*

The problem is:

Can m persons keep a watch on m days, n persons keeping a watch daily, so that any two persons go together only on two different days?

Evidently a solution can exist only if $m = \frac{n(n-1)}{2} + 1$.

I give below solutions for $n = 3, 4, 5$, and 6. It will be noticed that my solution for $n = 6$ is not elegant. I believe that a solution exists in the general case but I have not succeeded so far in obtaining it.

We denote the m persons by the numbers 1, 2, 3, ..., m.

(i) $n = 3$, $m = 4$.

The groups are:—

1, 2, 4;
2, 3, 1;
3, 4, 2;
4, 1, 3.

(ii) $n = 4$, $m - 7$.

The groups are:—

1, 2, 4, 7;
2, 3, 5, 1;
3, 4, 6, 2;
4, 5, 7, 3:
5, 6, 1, 4:
6, 7, 2, 5;
7, 1, 3, 6.

(iii) $n = 5$, $m = 11$.

The groups are:—

1, 2, 4, 7, 11;
2, 3, 5, 8, 1;
3, 4, 6, 9, 2;
4, 5, 7, 10, 3;
5, 6, 8, 11, 4:

---

[*] To me this problem was suggested by Dr. Chowla.

6, 7, 9, 1, 5;
7, 8, 10, 2, 6;
8, 9, 11, 3, 7;
9, 10, 1, 4, 8;
10, 11, 2, 5, 9;
11, 1, 3, 6, 10.

(iv) $n=6$, $m=16$.

The groups are :—

1, 2, 3, 4, 5, 6;
1, 2, 7, 8, 9, 10;
1, 3, 7, 11, 12, 13;
1, 4, 8, 11, 14, 15;
1, 5, 9, 12, 14, 16;
1, 6, 10, 13, 15, 16;
2, 3, 10, 12, 14, 15;
2, 4, 8, 12, 13, 16;
2, 5, 9, 11, 13, 15;
2, 6, 7, 11, 14, 16;
3, 4, 9, 10, 11, 16;
3, 5, 7, 8, 15, 16;
3, 6, 8, 9, 13, 14;
4, 5, 7, 10, 13, 14;
4, 6, 7, 9, 12, 15;
5, 6, 8, 10, 11, 12.

HANSRAJ GUPTA.

On the Converse of Fermat's Theorem

Fermat's theorem states that, if n is a prime and a is prime to n, then $a^{n-1}-1$ is a multiple of n. It is wellknown that the converse is not true. Mr. Kaprekar showed me a table of composite value of n for which $2^{n-1}-1$ is a multiple of n, in Sphinx. Let $F(x)$ denote the number of composite $n \leqslant x$, for which $2^{n-1}-1$ is a multiple of n. The object of this note is to prove

THEOREM II: $\lim F(x) = \infty$.

To prove the above result, we want

THEOREM I: *Let*

(1) a be an integer > 1;

(2) p an odd prime;

(3) $P_1, P_2, \cdots\cdots, P_k$ odd prime factors of $a^{2p}-1$, different from the prime factors of a^2-1 and p; and

(4) Q be the product of any r number $P_1, P_2, \cdots\cdots, P_k$ Then $a^{Q-1}-1$ is a multiple of Q.

From a theorem of Euler, we know that $P_1, P_2, \cdots\cdots, P_k$ are of the form $2mp+1$. So $Q=2Mp+1$. *Hence* $a^{p-1}-1$ *is a multiple of* $a^{2p}-1$ and so a multiple of Q.

When $p \geqq 5$, each of $2^{p-1}-1$ and $2^{p-1}+1$ should contain at least one prime of the form $2mp+1$. So corresponding to every $p > 5$, there should be at least one composite Q for which $2^{Q-1}-1$ should be a multiple of Q. Hence we get theorem II and

THEOREM III: $\underline{\lim}\ F(x) \log\log x/\log x > 0$.

The general principle can be stated as

THEOREM IV: If Q be the product of any number of prime factors of a^m-1 such that $Q=Mm+1$, then Q is a factor of $a^{Q-1}-1$.

Though theorem IV is too general to be of any use, it shows that theorem I will not lead to the evaluation of all composite n for which $2^{n-1}-1$ is a multiple of n.

e.g. for theorem I: $3^{10}-1=8\times 11\times 11\times 61$.

Here, $a=3$, $p=5$, $k=3$, $P_1=11$, $P_2=11$ and $P_3=61$.

So $Q=11^2$, 11×61 or $11^2\times 61$. Hence $3^{120}-1$, $3^{670}-1$ and $3^{7380}-1$ are multiples of 121, 671 and 7381 respectively.

Annamalainagar S. S. PILLAI.

On the Sum of $\sum\limits_{n=0}^{\infty} x^{12n}/(12n)!$

In a recent paper* Ludwig Silberstein has shown that

$$(1)\quad 1+\frac{x^3}{\underline{|3}}+\frac{x^6}{\underline{|6}}+\cdots\cdots=\frac{1}{3}\left\{e^x+2e^{-x/2}\cos\frac{\sqrt{3}}{2}x\right\}$$

$$(2)\quad 1-\frac{x^3}{\underline{|3}}+\frac{x^6}{\underline{|6}}-\cdots\cdots=\frac{1}{3}\left\{e^{-x}+2e^{x/2}\cos\frac{\sqrt{3}}{2}x\right\}$$

---

* *Phil Mag.* Vol. XXV page 1003 (1938).

$$(3)\quad 1+\frac{x^6}{\underline{|6}}+\frac{x^{12}}{\underline{|12}}+\cdots\cdots=\frac{1}{3}\left\{\cosh x+2\cosh\frac{x}{2}\cdot\cos\frac{\sqrt{3}}{2}\,x\right\}$$

We here sum two similar series.

The series (1), (2), (3) are evidently absolutely convergent for any finite x.

Replacing x by ix in (3), which is valid because of the convergence of (3) for all finite x (real or complex), we have since, $\cosh ix = \cos x$

$$(4)\quad 1-\frac{x^6}{\underline{|6}}+\frac{x^{12}}{\underline{|12}}-\cdots\cdots=\frac{1}{3}\left\{\cos x+2\cos\frac{x}{2}\cosh\frac{\sqrt{3}}{2}\,x\right\}$$

Taking half the sum of (3) and (4)

$$(5)\quad 1+\frac{x^{12}}{\underline{|12}}+\frac{x^{24}}{\underline{|24}}+\cdots\cdots=\frac{1}{6}\left\{\cosh x+\cos x+2\cosh\frac{x}{2}\cos\frac{\sqrt{3}}{2}x\right.$$
$$\left.+2\cos\frac{x}{2}\cosh\frac{\sqrt{3}}{2}\,x\right\}$$

Benares Hindu University S. SASTRY.

SOLUTIONS TO QUESTIONS

Question 1700

(V. Thibault, Le Mans, France): With the digits 0, 1, 2, . . . , 9 taken one at a time form a perfect square whose square root is of the form *abbba*.

Solution by D. K. Sen

This problem affords a good illustration of the methods of Practical Mathematics on which so much stress is nowadays laid in Europe and America during high school training. I shall give here a method of solution bearing on the principles of Practical Mathematics. In this method our aim is to obtain the solution in the simplest and shortest way. There will not be an academical discussion of the problem from the standpoint of Theory of Numbers or Analysis.

The problem is to find the square root of a number of ten digits wherein the figures 1, 2, 3, 4, 5, 6, 7. 8, 9 and 0 occur only once, the root being of the form *abbba*.

It is clear that *a* must be greater than 2, for otherwise the square of the number would consist of nine digits. This is also evident if we take their logarithms; the logarithm of the square being clearly greater than 9, that of the root must be greater than 4·5 showing that the number is more than 30000.

Further the sum of the digits of the square is clearly 45; therefore it is divisible by 9. Hence its square root should be divisible by 3; or the sum of its digits must be a multiple of 3. Therefore a can only have the values 3, 6 or 9 so that $2a+3b$ may be divisible by 3. The rule of casting out of nines then shows that $2a+3b$ should be equal to 3 or 12 or 21 or 30 or 39, etc. Hence we get the following combinations only:

$$\left.\begin{array}{l}a=3\\b=2\end{array}\right\}\left.\begin{array}{l}a=6\\b=0\end{array}\right\}\left.\begin{array}{l}a=6\\b=3\end{array}\right\}\left.\begin{array}{l}a=6\\b=9\end{array}\right\}\left.\begin{array}{l}a=9\\b=1\end{array}\right\}\left.\begin{array}{l}a=9\\b=4\end{array}\right\}\left.\begin{array}{l}a=9\\b=7\end{array}\right\}.$$

It is not very difficult to verify, by actual multiplication, that the last pair only gives the required square. Hence the number desired is 97779. As there are seven numbers to be tried the process should not be a prohibitive one. Even here one can shorten the labour of calculation. For example, let us take the fourth pair forming the number 69996 and try to write down a few steps of the multiplication similar to those of contracted multiplication, thus:

$$\begin{array}{r}
{*}{*}0016 \\ \hline
36 \\
108 \\
189 \\
{*}{*} \\
{*}\,{*}
\end{array}$$

There is no need to proceed further since we obtain in the sum (which is written out at the top instead of at the bottom for the sake of convenience) two 0s which is contrary to the given condition. The values of the various lines could be easily obtained from the following expression for the square in terms of a and b.

$$(a^2)+10.\,(2ab)+10^2.\,(2ab+b^2)+10^3.\,(2ab+2b^2)+10^4.\,(2a^2+3b^2)+\cdots\cdot$$

The values of the quantities in the brackets are written out in the successive lines.

Also solved by Messrs. V. K. Narasimhamurti, T. K. Raghavachari, K. Subba Rao and the proposer.

Question 1786

(K. S. K. Iyengar): Given a class $c(f)$ of functions $f(x)$ defined in $0 \leqslant x \leqslant 2\pi$ with the following properties:—

(1) f is L^2 in $(0, 2\pi)$.

(2) The Fourier series of f converges in the mean to f uniformly in the class $c(f)$.

(3) If f_1 and f_2 are any two functions of $c(f)$ then

$$\int_0^{2\pi}(f_1-f_2)^2\,dx \geq k \text{ where } k \text{ is a positive constant};$$

prove that:

(A) 1. The class $c(f)$ is enumerable;

2. If the sequence $\{f_n\} \equiv c(f)$ and $R_n = \left\{\int_0^{2\pi} f_n^2\,dt\right\}^{\frac{1}{2}}$, then the index of convergence of R_n is finite;

(B) If the class c consists of trigonometric polynomials of degree not greater than n_0 and satisfy condition (3) given in the beginning then the index of convergence of the corresponding R_n's is not greater than $2n_0+1$;

(C) A Theorem similar to (A) and (B) can be enunciated with regard to algebraic polynomials, the index of convergence being not greater than n_0+1.

Solution by the Proposer

§ 1. Let $\{\phi_n\}$ be a complete normal orthogonal set over a finite interval say $(0, 1)$ and let $\alpha_n = (f\phi_n) = \int_0^1 f\phi_n dx$ so that the series $\sum \alpha_n\phi_n$ converges in the mean to f.

We shall prove the results of the question for any complete normal orthogonal set taking the interval as $(0, 1)$.

§ 2. *Preliminary remarks and a subsidiary Theorem:—*

The problem should be viewed as one in Hilbert space; there exists a $1-1$ correspondence between functions of the class $c(f)$ and the set of points G(P) in Hilbert space with coordinates given by $P = (\alpha_1, \alpha_2 \cdots \alpha_n \cdots)$. The set of points G(P) being isolated (condition 3) and Hilbert space being separable, conclusion A(1) follows at once. However for the benefit of those not conversant with the terminology of Hilbert space, a direct proof of this result will be given later. Before proceeding to prove the main theorem, we shall first prove a subsidiary theorem which will be used in our arguments later.

THEOREM I: If G be a set of points in Euclidean p-space such that the distance PQ between any two points of G is $> k$, a fixed positive number; then

(1) G(P) is enumerable

(2) If $G \equiv \{P_n\}$ and O a fixed point, then the index of convergence of OP_n is not greater than p.

Proof: Because of the condition $PQ > k$, it is clear that in any finite p-sphere with O as centre, there will be only a finite number of points of G so that the points p of G can be so enumerated that

$$OP_1 < OP_2 \cdots < OP_n \longrightarrow \infty \text{ as } n \longrightarrow \infty \cdots \qquad (2{\cdot}1)$$

With $P_1, P_2 \cdots P_n$ as centres draw n spheres of radius $k/2$ then it is clear that none of these spheres overlap and they all lie in the sphere with centre O and radius $OP_n + k/2$, so that

$$n\left(\frac{k}{2}\right)^p < \left(OP_n + \frac{k}{2}\right)^p$$

$$\text{or} \qquad OP_n > \frac{k}{2}\left\{n^{1/p} - 1\right\} \cdot \cdot \qquad \ldots \quad (2{\cdot}2)$$

It is clear from this that $\sum OP_n^{-(p+a)}$ converges for $a > 0$. Hence the theorem.

§ 3. *Proof of enumerability of $c(f)$*:—

Corresponding to any f of $c(f)$ we have the orthogonal series

$$\sum a_n^f \phi_n \sim f, \text{ and } \int_0^1 f^2\, dx = \sum_0^\infty (a_n^f)^2 \cdot \cdot \cdot \cdot \qquad (3{\cdot}1)$$

Given $\epsilon > 0$, let $c_n(f)$ be the subclass of $c(f)$ such that for all f in it

$$\sum_{m=n+1}^{\infty} (a_m^f)^2 < \epsilon \cdot \cdot \cdot \qquad \ldots \quad (3{\cdot}2)$$

Since $\sum_1^\infty c_n(f) = c(f)$, the enumerability of $c(f)$ will follow from the enumerability of $c_n(f)$.

If f_1 and f_2 belong to $c_n(f)$, then

$$\int_0^1 (f_1 - f_2)^2\, dx = \sum_1^\infty (a_n^1 - a_n^2)^2 > k$$

$$\text{and } \left\{\sum_{n+1}^{\infty} (a_m^1 - a_m^2)^2\right\}^{1/2} \leqslant \left\{\sum_{n+1}^{\infty} (a_m^1)^2\right\}^{1/2} + \left\{\sum_{n+1}^{\infty} (a_m^2)^2\right\}^{1/2}$$

$$\leqslant 2\sqrt{\epsilon}, \text{ by } (3{\cdot}2)$$

Hence $\sum_{n+1}^{\infty} (a_m^1 - a_m^2)^2 \leqslant 4\varepsilon$

Taking $\varepsilon = k/5$, we have

$$\sum_{1}^{n} (a_n^1 - a_n^2)^2 > \frac{k}{5} = k_1 \quad \cdot \quad \cdot \quad \cdot \qquad \ldots \quad (3\cdot3)$$

Consider now the set of points G (P) in Euclidean n-space whose co-ordinates are given by $(a_1, a_2 \cdot \cdot a_n)$ where $a_1, \cdot \cdot \cdot a_n$ are the first n orthogonal coefficients of any f in $c_n(f)$. Then because of (3·3), there is a 1 - 1 correspondence between G (P) and $c_n(f)$. Also, on account of (3·3) by Theorem I, G (P) is enumerable and hence $c_n(f)$ is enumerable.

§ 4. *Proof of A* (2):—

The Hypothesis (2) implies that, given any $\varepsilon > 0$ there must be an n_0 such that for all $n > n_0$ and all f in C (f)

$$\int_0^1 \left(f - \sum_1^n a_n^f \phi_n\right)^2 dx < \varepsilon$$

i.e.
$$\sum_{n_0+1}^{\infty} \left(a_n^f\right)^2 \leqslant \varepsilon \qquad \ldots \quad (4\cdot1)$$

Using the terminology of §3, (4·1) implies

$$c(f) \equiv c_{n_0}(f)$$

Since there is a 1·1 correspondence between $C_{n_0}(f)$ and points P of G (P) with co-ordinates given by the 1st n_0 orthogonal coefficients of f in C (f) and since

$$PQ^2 = \sum_1^{n_0} \left(a_n^1 - a_n^2\right)^2 > k\ 5 \quad \left(\text{taking } \varepsilon = \frac{k}{5}\right),$$

by theorem I, $\sum \frac{1}{OP_f^{n_0+\alpha}}$ will be convergent for $\alpha > 0$ ... (4·2)

Since $\int_0^1 f^2\, dx \geqslant \sum_1^{n_0} (a_n^f)^2 = (OP_f)^2$,

A (2) follows at once ... (4·3)

[NOTE: Utmost one point P of G (P) may coincide with O because of condition (3) and in $\sum \frac{1}{(OP_f)^{n_0+\alpha}}$ that term may be omitted.

§ 5. *Proof of* (C) :—

If all f of $c(f)$ are defined by $f = \sum_{1}^{n_0} \alpha_n \phi_n$, $c(f)$ now corresponds to points in Euclidean n_0-space. Hypothesis (B) and theorem I imply as in (4·3) that $\sum 1\,(R_f)^{n_0+\alpha}$ converges for $\alpha > 0$, where $R_f = \int_0^1 f^2 dx$.

... (5·1)

Taking for $\{\phi_n(x)\}$ the normal orthogonal algebraic polynomials in (0, 1), since by hypothesis the degree of f in $C(f)$ is not greater than n_0, f can be expressed in terms of the first n_0+1 $\phi^1{}_n$s and hence etc. Similar arguments will prove the conclusion about trigonometric polynomials.

§ 6. Although not connected with the problem directly, I have added some interesting applications of theorem I to a convergence problem :—

If $A_1, A_2 \cdot\cdot A_n$ be n independent vectors in Euclidean n-space (i.e. such that the determinant $\| A_1 A_2 \cdot\cdot A_n \| \neq 0$), and if $|N_1 A_1 + \cdot\cdot + N_n A_n|$ be the absolute value of the vector $N_1 A_1 + \cdots + N_n A_n$ then the multiple series

$$\sum_{N_n=-\infty}^{\infty} \cdots\cdots \sum_{N_1=-\infty}^{\infty} 1\, |N_1 A_1 + \cdots + N_n A_n|^{n+\alpha} \text{ is convergent for all}$$

$\alpha > 0$, where we omit $N_1 = N_2 = \cdots = N_n = 0$.

Proof: Let the end-point of the vector $N_1 A_1 + \cdot + N_n A_n$ be P and that of $N_1' A_1 + N_2' A_2 + \cdots$ be Q

Then $PQ = |(N_1 - N_1')A_1 + \cdot + (N_n - N_n')A_n| = |N_1'' A_1 + \cdot\cdot + N_n'' A_n|$ where all the N''^1s are not simultaneously zero.

It is easily proved that the hypothesis $\| A_1 \cdot\cdot A \|_n \neq 0$ implies that the minimum value of $|N_1 A_1 + \cdot\cdot + N_n A_n|$ is a positive number, where $N_1\ N_2 \cdot\cdot N_n$ can take all positive and negative integral values except $N_1 = N_2 = \cdot\cdot = N_n = 0$., so that

$PQ \geq \rho$ and by theorem A we get

$\sum \cdots \sum 1\, |N_1 A_1 + \cdots + N_n A_n|^{n+\alpha}$ converges for $\alpha > 0$.

A particular case of this type of series with which Honours students are familiar is $\sum \sum 1/|m_1 \omega_1 + m_2 \omega_2|^s$ where the determinantal condition reduces to the usual form: imaginary part of $\omega_2/\omega_1 \neq 0$.

ANNOUNCEMENTS AND NEWS

1. The following gentlemen have been elected members of the Society :—

Som Datt Chopra Esq., M.A., Prof. of Mathematics, D. A. V. College, Hoshaiarpur.

P. L. Bhatnagar Esq., M.Sc., Lecturer, St. Stephen's College, Delhi.

2. The committee of the Indian Mathematical Society has been reconstituted as follows :—

President: Dr. R. Vaidyanathaswami, M.A., D.Sc., F.R.S.E., Reader, University of Madras.

Treasurer: Prof. L. N. Subramanian, M.A., Madras Christian College.

Librarian: Prof. R. P. Shintre, M.A., Fergusson College, Poona.

Other Members:

1. The Hon'ble Dr. Sir Shah Sulaiman, Kt., LL.D., D.Sc., Allahabad.
2. Dr. F. W. Levi, Dr. Phil., Nat., Calcutta.
3. Dr. A. Narasinga Rao, M.A., L.T., D.Sc, Annamalainagar.
4. Rai Saheb Dr. P. L. Srivastava, M.A., D.Phil., Allahabad.
5. Dr. A. N. Singh, D.Sc., Lucknow.
6. Prof. D. D. Kosambi, M.A., Poona.
7. Prof. S. Mahadevan, M.A., L.T., Madras.
8. Dr. M. R. Siddiqi, M.A., Ph.D., Hyderabad (Deccan).
9. Dr. Ram Behari, M.A., Ph.D, Delhi.

The committee of the Society has placed on record its appreciation of the services rendered by Dr. R. P. Paranjpye who has been President of the Society since 1936, and by Prof. M. V. Arunachala Sastri who has been a member of the committee for several years.

Prof. M. T. Naraniengar, M.A. formerly Professor of Mathematics at the Central College, Bangalore, died at Bangalore on the 9th October 1940. Prof. Naraniengar was the Foundation Editor of the *Journal of the Indian Mathematical Society* and laboured unceasingly and almost singlehanded for two decades to make it a worthy embodiment of the creative activity of India in the field of higher Mathematics. He was elected President of the Society for the period 1930-32 and was universally beloved and respected for his high sense of duty and the simplicity and saintliness of his character. A special number of the Mathematics Student will soon be published dedicated to his memory. The Indian Mathematical Society has passed a resolution expressing its deep sense of sorrow at his death and recording its appreciation of his services to the Society.

Dr. S. Sivasankaranarayana Pillai of the Annamalai University has been appointed Reader in Mathematics in the Travancore University and is expected to take up his new duties in January 1941.

Dr. N. S. Nagendranath, D.Sc. who was awarded the 1851 Exhibition Fellowship for work on Neuclear Physics at Cambridge, has returned to India and has been appointed Reader in the Andhra University.

QUESTIONS FOR SOLUTION

1793. (D. R. KAPREKAR): Let $\{a_n\}$ be a sequence defined by $a_n = a_{n-1} + 3n$, a_0 being any positive integer. Prove that, irrespective of the value of a_0, the number

$$a_0 + 10a_1 + 10^2 a_2 + \cdot \cdot \cdot \; 10^n a_n + \cdot \cdot \cdot \cdot$$

if continued sufficiently far will contain the recurring digits

$$004\ 115\ 226\ 337\ 448\ 559\ 670\ 781\ 893$$

or any cyclic permutation of the above digits.

1794. (P. KESAVA MENON): Find the value of

$$\sqrt[3]{\left[\frac{3}{2} + \frac{1 \times 3}{2^2} \sqrt[3]{\left\{\frac{5}{2} + \frac{3 \times 5}{2^2} \sqrt[3]{\left(\frac{7}{2} + \frac{5 \times 7}{2^2} \times \cdot \cdot \cdot \; to \; \infty\right)}\right\}}\right]}$$

1795. (K. BALACHANDRAN)

$$\text{If } \Delta(r) = \begin{vmatrix} 1 & 1 & 0 & 0 & \ldots & 0 \\ \frac{1}{3!} & \frac{1}{2!} & 1 & 0 & \ldots & 0 \\ \frac{1}{5!} & \frac{1}{4!} & \frac{1}{2!} & 1 & \ldots & 0 \\ \frac{1}{7!} & \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{1}{(2r-1)!} & \frac{1}{(2r-2)!} & \ldots & \ldots & \ldots & \frac{1}{2!} \end{vmatrix}$$

and $(r > 1)$ show that $(2r-1)!\,\Delta(r)$ is an integer.

1796. (M. A. SRIRANGANARAYANA): Ox, Oy are two great circles of a sphere at right angles to each other. From any point P on the sphere perpendicular arcs PM, PN are drawn to Ox and Oy respectively. If $OM = x$, $ON = y$ be the coordinates of P, show that the equation to any great circle may be written in the form $A \tan x + B \tan y + C = 0$ and obtain the special forms

(i) $\cos \alpha \tan x + \sin \alpha \tan y = \tan p$

(ii) $\dfrac{\tan x}{\tan a} + \dfrac{\tan y}{\tan b} = 1$

(iii) $\tan y = \sqrt{(m^2 - \tan^2 c)} \tan x + \tan c$

1797. (A. A. KRISHNASWAMI AYYANGAR): Two conics S, S′ have vanishing θ, θ'. From any point on their real common chord, a tangent is drawn to meet S in A and a secant to cut S′ in B, C. Shew that S′ and the conic through A, B, C having three-pointic contact with S at A are such that two of their common tangents meet on a common chord.

GLEANING

e in all its glory

What a quantity of abstruse science for a bit of string! Let us not be surprised. A pellet of shot swinging at the end of a thread, a drop of dew trickling down a straw, a splash of water rippling under the kisses of the air, a mere trifle, after all, requires a titanic scaffolding, when we wish to examine it with the eye of calculation. We need the club of Hercules to crush a fly.

Our methods of mathematical investigation are certainly ingenious; we cannot too much admire the mighty brains that have invented them; but how slow and laborious they appear when compared with the smallest actualities! Will it never be given to probe reality in a simpler fashion? Will our intelligence be able one day to dispense with the heavy arsenal of formula? Why not?

Here we have the abracadabric number *e* reappearing, inscribed on a spider's thread. Let us examine, on a misty morning the meshwork that has been constructed during the night. Owing to their hygrometrical nature, the sticky threads are laden with tiny drops, and, bending under the burden, have become so many catenaries, so many chaplets arranged in exquisite order and following the curve of a swing. If the sun pierce the mist, the whole lights up with iridescent fires and becomes a resplendent cluster of diamonds. The number *e* is in its glory.

Geometry, that is to say, the science of harmony in space, presides over everything. We find it in the arrangement of the scales of a fir-cone, as in the arrangement of an Epeira's lime-snare; we find in the spiral of a snail-shell, in the chaplet of a spider's thread, as in the orbit of a planet; it is everywhere, as perfect in the world of atoms as in the world of immensities.

And this universal geometry tells us of an Universal Geometrician, whose divine compasses have measured all things. . . . . . . I prefer that, as an explanation of the logarithmic curve of the Ammonite and the Epeira, to the worm screwing up the tip of its tail. It may not perhaps be in accordance with latterday teaching, but it takes a loftier flight.

HENRI FABRE,
from L. G. Simon's *Fabre and Mathematics and other Essays.*

THE MATHEMATICS STUDENT

Volume VIII] DECEMBER 1940 [Number 4

ON A PARTICULAR INSCRIBED RECTANGLE OF A CONIC [*]

BY

S. CHIDAMBARAM, *Madras University.*

It is well-known that the inscribed rectangles of a conic play an important part in the theory of its normals.[†] In the binary representation of the conic the inscribed rectangles are given by the syzygetic pencil of quartics determined by the remarkable sextic representing the extremities of the axes and the points at infinity. The particular inscribed rectangle whose vertices are the four points of contact of the circular tangents, or alternatively whose sides are the four directrices, is of special interest. The object of this paper is to exhibit the significance of this rectangle in the metrical geometry of the conic.

1. The conic and its evolute

Since the evolute is a curve of order six and class four, there are twelve centres of curvature on the conic, and eight common tangents of the conic and its evolute. But a tangent to the evolute is a normal to the conic, and if it is also a tangent to the conic it must be a circular tangent. But there are only four circular tangents to the conic and hence the eight common tangents of the conic and its evolute coincide in pairs with the four circular tangents. From this, since the evolute is the envelope of the normals, it follows that the evolute touches the conic at the points of contact of the circular tangents. Therefore eight of the twelve centres of curvature which lie on the conic coincide in pairs with the vertices of the special rectangle we are considering. The other four centres of curvature which lie on the conic may be called the residual centres of curvature. These and their bases on the conic have been specified below, both metrically and in terms of the covariants of the special rectangle under consideration.

---

[*] My thanks are due to Dr. Vaidyanathaswami who has helped me much in the preparation of this paper.

[†] Vaidyanathaswami, "On the feet of concurrent normals of a conic." *Jour. Ind. Math. Soc.,* Vol. 18 (1930-31) page 296.

2. The normal correspondence

Let $f = a_x^4$ represent parametrically the points of contact of the circular tangents to a conic S. Then from Meyer's "transference principle" the ∞^3 conics outpolar to the confocal system generated by S cut out on S tetrads which are apolar to f. These ∞^3 conics are rectangular hyperbolas, since the circular points constitute one of the confocals of S. Hence,

The necessary and sufficient condition that four points t_1 t_2, t_3. t_4 on S should lie on an outpolar rectangular hyperbola is that

$$a_{t_1}, a_{t_2}, a_{t_3}, a_{t_4} = 0 \qquad \ldots \quad (2{\cdot}1)$$

In particular the tangent and normal at a point being conjugate lines at the right angles constitute an outpolar rectangular hyperbola. Hence if the normal at x meets the conic again in y we have from (2·1) $a_x^3 a_y = 0$. i.e. *the normal at a point x meets the conic again at the linear polar of x with respect to f.*

3. The Hessian of f

It follows from the last result above that the three normals which pass through a point y on the conic (other than that at y) are given by $a_y a_x^3 = 0$, i.e. by the first polar of y with respect to f. By varying y we obtain a pencil of cubics each of which gives three points on the conic such that the normals thereat concur at a point on the conic. If a member of this pencil has a double factor we get two consecutive normals intersecting on the conic and hence the double factor gives a point the centre of curvature at which lies on the conic. But the double factors are the factors of the Jacobian of the pencil which in this case is the hessian of f. Hence

The bases of the four residual centres of curvature are the vertices of the rectangle given by the hessian of f.

This theorem may also be stated as follows, from the known relation between a tetrad and its hessian on the conic.

Let Γ be the unique inpolar confocal to S. Then Γ intersects S in a rectangle whose vertices are precisely the bases of the four residual centres of curvature on the conic. (3·1)

4. The Steinerian of f.

To find the four residual centres of curvature on the conic, it may be observed that if y is one of them, the first polar of y with respect to f has a double factor, i.e. its discriminant is zero. The condition for this is

$$(ab)^2(cd)^2(ac)(bd)a_y b_y c_y d_y = 0.$$

This reduces to $\frac{1}{3} jf - \frac{1}{2} i\mathrm{H} = 0$; which is called the Steinerian of f. Hence

The Steinerian of f gives the inscribed rectangle of the conic, whose vertices are the four residual centres of curvature on the conic.

A metrical specification of the Steinerian points is given in § 6.

5. The Cohessian of f

First, the following theorem may be proved:—

If f is any binary quartic and four points ultimately coincident with a point x are equicross with their linear polars with respect to f, then x is a zero of the co-hessian of f*

Proof:—If y_1, y_2, y_3, y_4 be equicross with their linear polars with respect to $f = a_x^4 = b_x^4 = c_x^4 = d_x^4$, we have

$$\frac{(y_1 y_2)(y_3 y_4)}{(y_1 y_4)(y_3 y_2)} = \frac{(ab)\, a_{y_1}^3\, b_{y_2}^3 \cdot (cd)\, c_{y_3}^3\, d_{y_4}^3}{(ad)\, a_{y_1}^3\, d_{y_4}^3 \cdot (cb)\, c_{y_3}^3\, b_{y_2}^3} \quad \ldots \quad (1)$$

But by Gordan's Theorem,

$$(ab)\, a_{y_1}^3\, b_{y_2}^3 = \{\tfrac{3}{2}\, [(ab)^2\, a_{y_1}^2\, b_{y_1}^2]_{y_2^2} + \tfrac{1}{4}\, i\, (y_1 y_2)^2\}\, (y_1 y_2)$$

Expanding similarly all the terms on the right hand side of (1) and cancelling common factors on both sides, the equation (1) becomes

$$\{\tfrac{3}{2}\, \mathrm{H}_{y_1^2 y_2^2} + \tfrac{1}{4}\, i\, (y_1 y_2)^2\}\{\tfrac{3}{2}\, \mathrm{H}_{y_3^2 y_4^2} + \tfrac{1}{4}\, i\, (y_3 y_4)^2\} -$$
$$\{\tfrac{3}{2}\, \mathrm{H}_{y_1^2 y_4^2} + \tfrac{1}{4}\, i\, (y_1 y_4)^2\}\{\tfrac{3}{2}\, \mathrm{H}_{y_3^2 y_2^2} + \tfrac{1}{4}\, i\, (y_3 y_2)^2\} = 0$$

where H is the hessian of f.

Putting $y_1 = y_2 = x$ and $y_3 = y_4 = y$ in this we get

$$\tfrac{9}{4}\, \mathrm{H}_x^4\, \mathrm{H}_y^4 - \{\tfrac{3}{2}\, \mathrm{H}_{x^2 y^2} + \tfrac{1}{4}\, i\, (xy)^2\}^2 = 0 \quad \ldots \quad (2)$$

The left-hand member of (2) is a double binary form in x and y and may be expanded in powers of (xy), i.e. in the form

$$\mathrm{A} + \mathrm{B}\, (xy) + \mathrm{C}\, (xy)^2 + \cdots$$

---

* If f is a binary n-ic, the corresponding covariant is $2\,(n-2)\, jf - ni\, \mathrm{R}$, where j is the second wronskian, i the fourth transvectant and h the hessian of f. The method of proof is the same as for the quartic.

Now $A=0$, since the expression vanishes when $x=y$. Also the expression is symmetrical with respect to x and y and therefore $B=0$. If C is not zero we may divide out the expression by $(xy)^2$ which is not zero and the quotient will tend to C as $y \rightarrow x$. But C is the same but for a numerical factor, as the result got by operating on the expression with Ω^2 (Ω being the differential operator $\left(\frac{\partial^2}{\partial x_1 \partial y_2} - \frac{\partial^2}{\partial x_2 \partial y_1}\right)$ and then putting $y=x$ in the resulting expression. That result is found on reduction to be $126\, jf - 126\, i\mathrm{H}$, where j and i are the two invariants of f. Therefore the required covariant is $jf - i\mathrm{H}$, which is the co-hessian of f.

Now we have already proved that the normal at a point x intersects the conic again in the linear polar of x with respect to f. Also the lines joining corresponding points of two homographic ranges on the conic envelope a conic having double contact with the given conic. It is also known that the cohessian is the only member of the syzygetic pencil apolar to f and that it has the same hessian as that of f. Putting these facts together we may derive the following theorem.

Let Γ' be the inpolar conic through the hessian points, other than Γ of (3·1.) Then the contact tetrad of Γ' with S (viz the cohessian tetrad of f) has the properties :—

(1) It is the unique inscribed rectangle of S, such that an outpolar rectangular hyperbola can be drawn through its vertices.

(2) Each point P of the tetrad is such that there is a conic having double contact with S and having four point contact with the evolute of S at the centre of curvature at P.

6. The apolar triads of f

Since f is a rectangle an apolar triad of f is a pedal triad (i.e. the normals at the three points are concurrent *).

Further from 2·1 PQR is a triad apolar to f if and only if every outpolar conic through P, Q, R is a rectangular hyperbola, or what is the same thing, if the centre of perspective of PQR and its polar triangle is the orthocentre of PQR.

Now the *triangles apolar to f form a pencil of inscribed triangles* in S and therefore circumscribe a unique conic S′. A particular triangle apolar to f is one whose vertices are two coincident points at a hessian point and the corresponding steinerian point, for this triangle is such that the line joining each vertex to the pole of the opposite side is perpendicular to that side. Therefore the conic S′ touches the tangents and normals at the hessian points and also touches the evolute of S at the steinerian points. But the tangents and normals to S at the hessian points are respectively the tangents to S and Γ, at their points of intersection, (since confocal conics cut at right angles). Therefore S′ is none other than the ϕ-conic of S and Γ.

---

* Vaidyanathaswami loc. cit.

Incidentally we have proved the theorem :

The four residual centres of curvature on the conic S, namely the steinerian points of f, are the points of intersection of S and the ϕ-conic of S and its inpolar confocal.

7. A (2, 2) Correspondence on the conic

It is a consequence of (2 1) that the necessary and sufficient condition that an outpolar rectangular hyperbola should touch the conic S at the two points x and y is that $a_x^2 a_y^2 = 0$.

Now the polar correspondence given by $a_x^2 a_y^2$ is the correspondence determined by the polarity of the unique inpolar confocal Γ, considered as an envelope.

Also the discriminant of $a_x^2 a_y^2$ considered as a quadratic is x or y in the hessian of f.

Hence we have the following theorem.

Given a point P on the conic S, there are two outpolar rectangular hyperbolas touching S at P and having double contact with S, their other points of contact with S being the points of contact of the two tangents of S which are conjugate with regard to Γ, to the tangent at P to S. When P is the point of contact of a circular tangent, the two outpolar rectangular hyperbolas coincide with the circular tangent taken twice. When P is a residual centre of curvature the two outpolar rectangular hyperbolas coincide into one touching S at P and its base on the conic.

THE LIMITING POINTS OF THE SET OF THE UPPER AND THE LOWER SUMS IN RIEMANN INTEGRATION

BY

B. S. SASTRY, *Central College, Bangalore.*

§ 1. $f(x)$ being a function bounded in $a<x<b$, the set E of the upper sums $\sum M_r\delta_r$, and e of the lower sums $\sum m_r\delta_r$, for all modes of subdividing (a, b) into abutting subintervals, are bounded sets.* In this paper I consider the limiting points of these sets and describe the mode of subdividing (a, b) so as to give an upper sum (or a lower sum as the case may be), which differs from a given limiting point by an arbitrarily assigned positive number ε. The lower limit U of the set E and the upper limit L of the set e are considered in Darboux's theorem, which gives subdivisions of (a, b), whose upper sums are $<U+\varepsilon$ and lower sums $>L-\varepsilon$; to get such sums the mode of subdivision is merely to make the maximum sub-interval less than a certain η.

§ 2. THEOREM : *If M and m be the upper and the lower bounds, U and L the upper and the lower Integrals of $f(x)$ in (a, b), then every point between U and M $(b-a)$ is a limiting point of the set E of the upper sums; and similarly, every point between L and m $(b-a)$ is a limiting point of the set e of the lower sums.*

Proof: Let P be a number between U and Mh and $P=Kh$, where $h=(b-a)$. Now there exists a number μ, $M\geqslant\mu\geqslant m$, such that $\mu h=U$. Clearly $M>K\geqslant\mu\geqslant m$. Consider $M(x-a)-\overline{\int_a^x} f(x)dx$. Since this is a continuous function of x, whose maximum and minimum values in (a, b) are $(M-\mu)h$ and zero, therefore there exists a value c of x such that

$$M(c-a)-\overline{\int_a^c} f(x)dx=(K-\mu)h.$$

Let $\int_a^c f(x)dx=A$. Then $\int_c^b f(x)dx=\mu h-A$. Now (it is easily proved by the method of continued bisection that) there always exists a point ξ in (a, b) such that every neighbourhood of ξ has M for upper bound.

First let us prove our theorem taking ξ to coincide with a. Then in any subdivision of (a, b) in which (a, c) is a sub-interval, the contribution of (a, c) to the upper sum is equal to $A+(K-\mu)h$. And given $\varepsilon>0$ arbitrarily, we can, by Darboux's theorem, find an $\eta>0$ such that when (c, b) is subdivided into intervals of length less than η each, the upper sum lies between $\mu h-A$ and $\mu h-A+\varepsilon$. Every such subdivision of (c, b) together with the interval (a, c) will therefore constitute a sub-division of (a, b) whose upper sum lies between $A+(K-\mu)h+(\mu h-A)$ and $A+(K-\mu)h+(\mu h-A)+\varepsilon$ i.e. between P and $P+\varepsilon$.

---

* M_r and m_r are the upper and the lower bounds of $f(x)$ in the sub-interval δ_r.

But (c, b) can be sub-divided in an infinity of ways such that the maximum sub-interval is less than η, and we have only to add the interval (a, c) to each of these subdivisions of (c, b) to obtain a subdivision of (a, b). Therefore there exist an infinity of upper sums between P and $P+\varepsilon$. Hence P is a limiting point of the set E.

We have taken $a=\xi$ in the above proof for the sake of convenience. But even if $\xi\neq a$, we can, since the upper integral is a continuous function of the upper limit of integration, still obtain an interval (a_1, c_1) in (a, b) so that ξ is an interior point (or end point according to circumstances) of (a_1, c_1) and at the same time

$$M(c_1-a_1)-\int_{a_1}^{c_1} f(x)dx=(K-\mu)h.$$

Then in the above proof we have only to replace the interval (a, c) by the interval (a_1, c_1) and (c, b) by the intervals (a, a_1), (c_1, b).

In the case of the set e we should take $\int_a^x f(x)dx-m(x-a)$ instead of $M(x-a)-\int_a^x f(x)dx$ and the interval (a_1, c_1) so that the lower bound of $f(x)$ in it is m.

§ 3. Consider for instance the function $f(x)$ defined in (0,1) by $f(x)=x$ when x is irrational, and $f(x)=\left(\frac{1+p^2}{1+q^2}\right)^{\frac{1}{2}}$ when $x=\frac{p}{q}$, and $f(0)=0$. The discontinuities of this function are enumerable since they occur at rational points (except 0 and 1*). The function is therefore integrable and the value of the integral is easily seen to be $\frac{1}{2}$ by a cosideration of the upper or the lower sums. We have $h=1$, $\mu=\frac{1}{2}$, $U=L=\frac{1}{2}$, $M=1$, $m=0$, $\xi=1$ (M is attained at ξ). All points between $\frac{1}{2}$ and 1 are limiting points of E and all points between 0 and $\frac{1}{2}$ are limiting points of e. For, suppose we wanted a subdivision of (0,1) whose upper sum differs from the number $\frac{3}{4}$ by a given quantity. Then $(K-\mu)\,h=\frac{1}{4}$ and since $f(1)=M$, therefore we must find a point a_1 such that $\int_{a_1}^{1}[1-f(x)]\,dx=\frac{1}{4}$. But $\int_{a_1}^{1} f(x)\,dx=(1-a_1)(1+a_1)/2$. Hence

---

* The continuity at $x=1$ is omitted in Hardy's *Pure Mathematics* Ex. 21. P. 179, 4th Edition.

a_1 should satisfy the equation $2a_1^2 - 4a_1 + 1 = 0$. So, $a_1 = \left(1 - \frac{1}{\sqrt{2}}\right)$. Thus (a_1, c_1) is the interval $\left(1 - \frac{1}{\sqrt{2}}, 1\right)$. To subdivide $\left(0, 1 - \frac{1}{\sqrt{2}}\right)$ we first observe that $\left(\frac{1+p^2}{1+q^2}\right)^{\frac{1}{2}} - \frac{p}{q} < \varepsilon$ i. e.

$$q^2\ (\varepsilon^2\ q^2 + \varepsilon^2 - 1) + p\ (2\varepsilon\ q^3 + 2\varepsilon q + p) > 0,$$

if $(\varepsilon^2 q^2 + \varepsilon^2 - 1) > 0$ (for all $p < q$); and $(\varepsilon^2 q^2 + \varepsilon^2 - 1) > 0$ if $q > 1/\varepsilon$. Thus when $q > 1/\varepsilon$, we have $\left(\frac{1+p^2}{1+q^2}\right)^{\frac{1}{2}} - \frac{p}{q} < \varepsilon$.

Now divide $\left(0, 1 - \frac{1}{\sqrt{2}}\right)$ into n^3 intervals of length $\frac{1}{n^3}\left(1 - \frac{1}{\sqrt{2}}\right)$ each, where $n > 1/\varepsilon$. Mark off the points for which $n \leqslant 1\ \varepsilon$. Such points are not more than n^2 in number and are situated in not more than n^2 of the n^3 sub-intervals. The contribution S_1 of the n^2 intervals to the upper sum is less than

$$n^2 .\ 1 .\ \frac{1}{n^3}\left(1 - \frac{1}{\sqrt{2}}\right) < \left(1 - \frac{1}{\sqrt{2}}\right)\varepsilon, \qquad \ldots \quad \text{(i)}$$

since the maximum of $f(x)$ is $\leqslant 1$. The contribution S_2 of the remaining $(n^3 - n^2)$ intervals is less than

$$\frac{1}{n^3}\left(1 - \frac{1}{\sqrt{2}}\right)\sum_{\gamma=1}^{n^3}\left[\frac{\gamma\left(1 - \frac{1}{\sqrt{2}}\right)}{n^3} + \varepsilon\right]; \qquad \ldots \quad \text{(ii)}$$

for, in the interval $\left[\frac{(\gamma - 1)\left(1 - \frac{1}{\sqrt{2}}\right)}{n^3}, \frac{\gamma\left(1 - \frac{1}{\sqrt{2}}\right)}{n^3}\right]$ the maximum of $f(x)$ is less than $\left[\frac{\gamma\left(1 - \frac{1}{\sqrt{2}}\right)}{n^3} + \varepsilon\right]$. In forming (ii) we have, for the sake of convenience, repeated the n^2 intervals each multiplied by the corresponding $\frac{\gamma\left(1 - \frac{1}{\sqrt{2}}\right)}{n^3} + \varepsilon$. The effect has been merely to increase the quantity than which S_2 is less.

Now the left member of (ii) is equal to

$$\left(1 - \frac{1}{\sqrt{2}}\right)^2\left(\varepsilon + \frac{n^3 + 1}{2n^3}\right) < \left(1 - \frac{1}{\sqrt{2}}\right)^2\left(\tfrac{1}{2} + \varepsilon + \frac{\varepsilon^3}{2}\right).$$

Hence from (i), $S_1 + S_2 < \frac{3}{4} + \varepsilon$. Similarly we could take any number between 0 and $\frac{1}{2}$ and show it to be a limiting point of the set e.

A NOTE ON FREQUENCY DISTRIBUTION IN SERIES

By

D. D. Kosambi.

For a random variable that can only assume discrete values. the method of generating functions is in common use as an aid to formulating moments.[1] The probability of the value x_j being p_j, the probability generating function is defined by $\sum p_j z^{x_j}$, and moments, factorial moments, and semi-invariants are obtained as coefficients in the various power-series expansions if they exist, of the function after substitution of e^a, $1+a$, for the parameter z, the third being obtained by expanding the logarithm of the function after the first change of variable. But here, the parameter is introduced "for the sole purpose of preventing the terms from being merged together",[2] and is, so to speak, without any value of its own.

I suggest, in what follows, a similar method of generating the probabilities and the moments, but making use of the variable z as an essential part of the distribution. The difference is that one obtains formulae involving the differential calculus in place of algebra, and a connection is to be seen at once with the formal apparatus of the theory of functions of a complex variable, summability, and Tauberian theorems.

1. Let $f(z)$ be an analytic function in the neighbourhood of the origin, with real non-negative derivatives at the origin. That is, $f(z)=a_0+a_1 z+a_2 z^2+a_3 z^3+\cdots+a_n z^n+\cdots$, a_i real, $\geqslant 0$ for all i, and let the series converge to f for some region beyond just the origin itself, say $|z|<r$. Then, we shall say that f gives a distribution, the probability of the value n being given by $a_n z^n/f$. For the present, the variate x is assumed to take on only the values $0, 1, 2, 3, \cdots n \cdots$

The kth moment, the expectation of n^k, is given by $\dfrac{\theta^k f}{f}$, where θ is the operator $z\dfrac{d}{dz}$ and the exponent k indicates a k-fold application. The corresponding factorial moment is given by $\dfrac{z^k\, d^k f}{f\, dz^k}$. The operator θ and the occurrence of the factor $1/f$ suggest immediately, the transformation $\phi = \log f$, $s = \log z$. Then the successive derivatives of $\phi(s)$ are the semi-invariants λ_i; $\phi' = m$, $\phi'' = \sigma^2$ etc.

By merely specifying the function, various distributions can be characterized. The Bernoulli distribution is given by $f = [1+z]^n$. In the usual terminology, $p = z/(1+z)$, $q = 1/(1+z)$ gives the probabilities for success and failure in a single trial. The mean value is $m = zf'/f = nz/(1+z)\ 1 = np$, the variance being $\sigma^2 = nz/(1+z)^2 = npq$. Here, as in general, we shall take only real non-negative values of z for statistical purposes.

For the Poisson distribution, $f = e^z$, $m = \sigma^2 = \lambda_i = z$. Distributions having various properties can be built up by solving the corresponding differential equations. But the use of $\phi(s)$ shows that a non-vanishing multiplicative arbitrary constant in f as well as in z can be absorbed, or ignored, without any essential change. Thus, $\sigma^2 = km$ leads to $f = e^{z^k}$. For $\sigma^2 = m - k$ we have $f = z^k e^z$. The first would correspond to a Poisson distribution of twins, triplets, etc. for $k = 2, 3, \ldots$; the second is essentially the generalised Poisson distribution for a variate that cannot take any value less than k. But a variate occurring in even or odd values only where m tends to equality with σ^2 as both increase would be distributed, say, according to $\cosh z$, or $\sinh z$.

2. The direct determination of f would be useful for some properties. The distribution of a sum of two variates distributed according to $A(z) = a_0 + a_1 z + \cdot\cdot$ and $B(t) = b_0 + b_1 t + \cdot\cdot$ would be given by the probability for the value n of the sum equal to $\sum_{i+j=n} a_i z^i b_j t^j / AB$. For two Bernoulli distributions with the same n the function is $(1 + z + t + zt)^n$, corresponding to a new parameter $z + t + zt$, or to the formula for compound probability $p = p_1 + p_2 - p_1 p_2 = p_1 q_2 + p_2 q_1 + p_1 p_2$. The Poisson distribution is essentially the only one where the sum is distributed according to the same law with the new parameter a sum of the two original parameters: $f(z)f(t) = f(z+t)$.

In some observations, even for continuous distributions, the quantities ϕ and s are observed directly, not f and z. Fechner's law naturally comes to mind here. One example that suggests itself is that of photo-sensitometry[8], where the logarithm of the darkening is plotted against the logarithm of the exposure for a given emulsion and time of development. The "gamma" value usually published is merely the slope of the curve, i.e. $\phi'(s)$ for the central portion, where the curve has a point of inflection, $\phi''=0$, and approximates very closely to its tangent. According to our theory, this is a mean value, and in fact, if the darkening depends in a certain way upon the number of elementary particles affected (the grains of the emulsion), this interpretation would be entirely accurate. The curve of sensitivity usually published looks uncommonly like half a Gaussian curve of normal frequency, but this cannot be verified unless very accurate measurements are made in the range of solarization; these last have been neglected, because they appear to have no practical use in photography, the useful portion of the curve being only in the neighbourhood of the point of inflection.

The transformation to s, ϕ, suggests that for a denumerable set of discrete values of the variable, the proper generalisation of our distribution is not in the form of power series with fractional exponents, but Dirichlet series for f or ϕ. In the usual notation, these would be given, substituting $-s$ for s by $\sum a_n e^{-\lambda_n s}$. The connection with the theory of numbers is obvious, and need not be considered in detail here.

3. Another obvious connection with analysis is the notion of summability given by each of our distributions. The Poisson distribution, for instance, leads to Borel summability. Let us interpret n as the number of observations, or the number attached to an observation or measurement, of some quantity s; let s_1, s_2, s_3 $\cdots$ be the successive values obtained in this way. The probability of n observations (or the nth observation) having taken place being distributed in a Poisson series, the expectation of s is $\sum e^{-z} s_n z^n/n!$ The question: when are we justified in taking the limit of this as $z \longrightarrow \infty$ as the limit of s_n, amounts to the Tauberian problem: when does a sequence *summable-B converge?* *A better question would be*: when can we take the limiting expectation as limit of the arithmetic means, that is when is B-summability equivalent to (C, 1) summability? Inasmuch as, in practice, the limit can never be attained, the important question is not only the approach to a limit, but actually the behaviour and distribution in the vicinity of the limit. That is, the Tauberian idea must be generalized. In view of the fact that a connection is known between Tauberian theorems and probability [6] through the Fourier-Stieltjes integral, I shall merely point out the phenomenon here.

4. Of the infinite series given, there naturally exist two classes: those with a finite and those with an infinite radius of convergence. For the first, we can take the radius as unity, since

a factor in the parameter z makes no real difference. The second class contains polynomials and integral functions, and is more interesting in a way because the distributions can be made to tend to the normal, under certain restrictions. For a polynomial, .this is a matter of adjusting n, the degree $(=\lim.\ zf'/f)$ and the parameter z carefully, letting n tend to infinity. For integral functions of infinite order, there seems to be no way of getting the normal distribution as a limit, because m and σ^2 are not of comparable magnitudes as $z\rightarrow\infty$. When the limit σ^2/m exists, it is the order of the integral function and the distribution for positive finite order approaches the normal under fairly general restrictions on a_n for the types of functions we use. The reason is that $f(z)$ does not then differ by much, relatively, from the greatest term in its expansion,[4] and the values of $a_n z^n \; f$ taper off rapidly to zero on either side. Transferring the origin to the greatest term (relatively near the mean value, which would otherwise recede infinitely far from the origin) and changing the scale, it is easy to show, (at least when there are limited gaps in the coefficients and the coefficients themselves decrease monotonically,) that the distribution approaches a continuous one obeying the differential equation $dF/Fdx = -cx$, which characterizes our normal distributions.

Series with a unit radius of convergence might also be useful, as the Lambert series distributions again permit results of the theory of numbers to be translated bodily into statistics. But in general, it would seem impossible to get the normal distribution as a limiting case, as is shown for example by the behaviour of the Abel-summability function $f(z)=1/(1-z)$, $p_n=z^n\ (1-z)$, as $z\rightarrow 1$.

5. As an illustration of all the foregoing, we take a problem usually solved by other methods, and show that its fundamentals are made simpler to the understanding by use of probability. Arrange a large number of urns in serial order, labelled 1, 2, 3...n..., and let each urn contain as many balls as there are prime factors in its label-integer. Each factor is to be counted according to its multiplicity and unity may be taken as a prime only for the first urn. Now the exact probability of there being k balls in an urn chosen at random is not known, but an approximation is obtained from an asymptotic value for the number $\pi_k(x)$ of integers $\leqslant x$ containing exactly k prime factors, $\pi_k \backsim \dfrac{x}{\log x}\,\dfrac{(\log\log x)^{k-1}}{(k-1)\,!}$

Take the value of the probability (the ratio π_k/x) obtained from this as exact. This can be transformed by taking $z=\log\log x$, $\log x=e^z$. The distribution function is seen to be ze^z; every number has at least one prime factor, and *we have a modified Poisson distribution* as explained in section 1. The expectation of the number of balls in an urn is $\frac{\theta z e^z}{z e^z} = z+1$. Thus we should have the total number of prime factors of all integers $\leqslant x$ as $x(\log \log x + 1)$. Now the known [6] formula for this is $x(\log \log x + \mathrm{B}) + O(x)$. The $O(x)$ term is naturally due to the neglected remainder, as we have taken an approximation as the exact value, and summed to infinity instead of stopping at x terms of the series. The value of the constant B is a little less than unity, due not only to the two causes given above, but also to the fact that there is a linkage between the total number of urns, and the probability; the parameter z is not independent of x. As the value of B is known, we do not investigate this further.

REFERENCES:

1. A. C. Aitken: *Statistical Mathematics*, (Edinburgh, 1939) sections 7-10.
2. *Ibid.* p. 19.
3. J. M. Eder: *Rezepte, Tabellen u. Arbeitsvorschriften f. Photographie* (Halle, 1927) pp. 251-258.
4. E. C. Titchmarsh: *Theory of Functions* (Oxford, 1932) chap. VII, particularly p. 253.
5. N. Wiener: *The Fourier Integral* (Oxford, 1933).
6. G. H. Hardy and E. K. M. Wright *Introduction to the Theory of Numbers* (Oxford 1938) Th. 435, p. 355.

GLEANINGS

Die ganzen Zahlen hat Gott gemacht; Alles andere ist Menschenwerk.

KRONECKER in *Jahresbericht* Bd. 2.

A THEOREM ON CONGRUENCE

BY

P. KESAVA MENON, *University of Madras.*

1. In this paper is discussed a generalization of the theorem that if p be an odd prime, then

$$\sum_{n=1}^{p-1} n^{p-1} - p - (p-1)! \equiv 0, \pmod{p^2}$$

which I have established elsewhere.*

1·1 We shall make use of the following notation:—

$\phi(m)$ = number of numbers less than m and prime to it. For convenience of printing it will be sometimes contracted into ϕ.

$\sum(n;\ r;\ p_1,\ p_2,\ \dots\ p_n)$ = the sum of the products taken r at a time of the n numbers $p_1,\ p_2\ \dots\ p_n$;

$$\binom{n}{r} = \frac{n(n-1)\ \cdots\ (n-r+1)}{r!}; \quad \binom{\phi(m)}{r} = \phi_r$$

2. THEOREM 1. *If* $n_1,\ n_2,\ \dots\ n^{\phi(m)}$ *be a reduced residue system, mod m, then*

$$\sum\left(\phi(m);\ r;\ n_1^{\phi},\ n_2^{\phi},\ \dots\ n_{\phi}^{\phi}\right) + \phi_r\left(r-1 \pm r\cdot n_1 n_2 \dots n^{\phi}\right)$$

$$\equiv 0, \pmod{m^2}, \qquad \dots \quad (1)$$

where the positive sign is to be taken when $m=4$, p^n *or* $2p^n$, *p being an odd prime, and the negative sign when m has any other value.*

§ 2·2 The proof depends on the LEMMA: *If* $p_1,\ p_2,\ \dots\ p_s$ *be numbers prime to m, then*

$$(p_1\ p_2\ .\ p_s)^t - \sum(s;\ 1;\ p_1^t,\ p_2^t, \dots p_s^t) + s - 1 \equiv 0, \pmod{m^2}, \dots \qquad (2)$$

where t is a multiple of $\phi(m)$.

This lemma is a particular case of the following

---

* A Theorem on Congruence. *Jour. Indian Math. Society* Vol. II, (New series) No. 8.

THEOREM 2. *If* $p_1, p_2, \cdots p_s$ *be numbers prime to* m, *and* t *be a multiple of* $\phi(m)$, *then*

$$(p_1\, p_2 \cdots p_s)^t -$$
$$\sum(s\,;\, r-1\,;\, p_1^t, p_2^t, \cdots p_s^t) +$$
$$\binom{s-r+1}{1} \sum\left(s\,;\, r-2\,;\, p_1^t, \cdots p_s^t\right) -$$
$$\binom{s-r+2}{2} \sum\left(s\,;\, r-3\,;\, p_1^t, \cdots p_s^t\right) + \cdots$$
$$+(-)^r \binom{s-1}{r-1} \equiv 0 \pmod{m^r} \qquad \ldots \quad (3)$$

Proof † $(x-p_1^t)(x-p_2^t) \cdots (x-p_s^t)$

$$\equiv (x-1)^s - S_1 (x-1)^{s-1} + \cdots + (-)^{r-1} S_{r-1} (x-1)^{s-r+1} \pmod{m^r}$$

where S_q denotes $\sum(s\,;\, q\,;\, p_1^t-1,\, p_2^t-1, \cdots p_s^t-1)$ since $S_r, S_{r+1}, \cdots S_s$ are $\equiv 0 \pmod{m^r}$, by Euler's generalization of Fermat's theorem.

Putting $x=0$ in (4) we get

$$p_1^t\, p_2^t \cdots p_s^t \equiv (1+S_1+S_2+ \cdots + S_{r-1}) \qquad \pmod{m^r} \qquad \ldots \quad (5)$$

Now $1+S_1+S_2+ \cdots +S_{r-1} =$ coefficient x^{r-1} in the expansion in powers of x of $(1-x+xp_1^t)(1-x+xp_2^t) \cdots (1-x+xp_s^t)(1-x)^{\ 1}$

$=$ coefficient of x^{r-1} in

$$(1-x)^{s-1} + x(1-x)^{s-2} \sum(s\,;\, 1\,;\, p_1^t, \cdots p_s^t) + \cdots$$
$$-(-)^{r\ 1}\binom{s-1}{r-1} + (-)^{r-2}\binom{s-2}{r-2} \sum\left(s\,;\, 1\,;\, p_1^t, \cdots p_s^t) + \cdots \cdot\right)$$

Substituting this value in (5) we get the required result.

3. *Proof of Theorem* 1.

In (2) take the p's to be the products, r at a time, of the numbers $n_1, n_2, \ldots n_{\phi(m)}$.

Then we get

$$\left(n_1, n_2 \ldots n_{\phi(m)}\right)^{r\phi_r} - \sum\left(\phi(m)\,;\, r\,;\, n_1^{\phi(m)}, \ldots n_\phi^\phi\right) + \binom{\phi(m)}{r} - 1$$
$$\equiv 0, \pmod{m^2}.$$

---

† The theorem can also be proved by induction on s.

By Gauss's generalization of Wilson's theorem we have,

$$n_1\, n_2 \ldots n_{\phi(m)} \equiv \begin{cases} -1;\ m=4,\ p^n,\ 2p^n \\ +1;\ m \neq 4,\ p^n,\ 2p^n \end{cases};\ p \text{ being an odd prime.}$$

Let $n_1\, n_2 \ldots n_{\phi(m)} = \lambda m \mp 1$.

Then
$$\left(\lambda m \mp 1\right)^{r\phi_r} - \sum\left(\phi(m);\ r;\ n_1^{\phi(m)}, \ldots n_{\phi}^{\phi}\right) + \left(\phi_r(m)\right) - 1 \equiv 0, \pmod{m^2}$$

Therefore, noting that $\phi(m)$ is even if $m>2$, we get

$$1 \mp \lambda m^{\phi_r}\, r - \sum\left(\varphi(m);\ r;\ n_1^{\phi(m)}, \ldots n_{\phi}^{\phi}\right) + \phi_r - 1 \equiv 0, \pmod{m^2}.$$

i.e.
$$\sum\left(\phi(m);\ r;\ n_1^{\phi(m)}, \ldots n_{\phi}^{\phi}\right) + \phi_r\left[r - 1 \pm r \cdot n_1\, n_2 \ldots n_{\phi(m)}\right] \equiv 0, \pmod{m^2}.$$

COROLLARY

Putting $r=1$, *we get*

$$\sum_{r=1}^{\phi(m)} n_r^{\phi(m)} \pm \phi(m)\, n_1\, n_2 \cdots n_{\phi} \equiv 0 \pmod{m^2}$$

THEOREM 3. *If* x *be prime to* m,

$$x^{s\phi(m)} - \binom{s}{r-1} x^{(r-1)\phi(m)} + \binom{s}{r-2}\binom{s-r+1}{1} x^{(r-2)\phi(m)} - \ldots + (-)^r \binom{s-1}{r-1} \equiv 0, \pmod{m^r}$$

This is easily proved by putting $p_1 = p_2 = \cdots = p_s = x$, $t = \phi(m)$ in theorem 2.

GLEANINGS

SOUND

The Soul, desirous of speaking out its intention, excites the Mind and the Mind operates on the Vital Heat of the body by setting the Air in motion, and the Air remaining in the *Brahmagranthi* rises up and produces Sound, through the navel, the heart, the neck, the head and the face.

RAMAMATYA in *Suārameḷakalanidhi*, (16th century).

THE ROLE OF UNIT PARTIAL QUOTIENTS IN SOME CONTINUED FRACTIONS

BY

A. A. KRISHNASWAMI AYYANGAR, *Mysore*.

1. The unit partial fractions, if any, in a simple continued fraction (briefly S. C. F.) play a significant role. They form the link between the simple continued fraction and the several half regular continued fractions (briefly *h. r.* C. F's). By their annihilation†(1) we obtain the most rapidly convergent *h. r.* C. F's as well as those of shortest length when the C. F's are finite. Minkowsky's(2) diagonal C. F. development of a real number other than an integer or half an integer is also based on the annihilation, under certain circumstances, of special unit partial quotients, viz, those that immediately follow or precede a partial quotient greater than 1. In this connection, we may point out a property related to diagonal C. F's not probably noticed before:

'*A unit partial quotient is no longer 'special' or 'ausgezeichnete' when it is preceded and followed by a unit partial quotient.*'††

This shows that the first and the last in a set of successive unit partial quotients (in between two partial quotients greater than 1) have a special character distinct from that of the intervening ones. We shall see presently another context in which a similar phenomenon occurs. We propose to discuss the properties of a class of pure recurring S. C. F's with special reference to the unit partial quotients.

For convenience, the following conventions will be adopted:

$$(1)\quad \xi_0 = (a_0, a_1, a_2, \ldots a_v, \ldots) = a_0 + \frac{1}{a_1 +}\,\frac{1}{a_2 +}\;\cdots\;\frac{1}{a_v} + \cdots;$$

ξ_v is the $(v+1)$-th complete quotient and corresponds to the partial quotient a_v.

---

† This and other subsequent numbers in brackets appearing between consecutive lines relate to references at the end of the paper.

†† On p. 184 footnote, Perron(3) has pointed out that $b_{\lambda+1}=1$ $(\lambda>0)$ is a special partial quotient if $(0, 1+b_{\lambda+3}, b_{\lambda+4}, \ldots) \geqslant (0, b_\lambda, b_{\lambda-1}, \ldots b_1)$, when $b_{\lambda+2}=1$. We now remark that this inequality is not satisfied, if $b_\lambda=1$.

(2) $\overline{(a_0, a_1, a_2, \cdots a_r, 1_{[n]}, a_{r+n+1}, \cdots a_p)}$

$$= a_0 + \frac{1}{a_1} + \frac{1}{a_2 +} \cdots + \frac{1}{a_r +} \frac{1}{1 +} \frac{1}{1 +} \cdots \frac{1}{1 +} \frac{1}{a_{r+n+1} +} \cdots \frac{1}{a_p}$$

(a pure recurring S. C. F with n unit partial quotients between a_r and a_{r+n+1});

(3) A quadratic surd of the form $(P+\sqrt{R})/Q$ is said to be standard, when P, Q, $(R-P^2)/Q$ are integers having no common factor R being a non-square positive integer.

2. THEOREM I:—*If* $\xi_0=(P_0+\sqrt{R})/Q_0$ *a standard surd develops as a pure recurring simple continued fraction with a set of successive unit partial quotients preceded and followed by other partial quotients, then the denominators of the complete quotients corresponding to the unit partial quotients other than the first and the last of the set are less than* $\sqrt{R}$.

Proof:—Let $\xi_0 = \dfrac{P_0+\sqrt{R}}{Q_0} = (\overline{a_0, a_1, \cdots a_r, 1_{[n]}, a_{r+n+1}, \cdots a_p})$.

Then $\xi_{r+v} = \dfrac{P_{r+v}+\sqrt{R}}{Q_{r+v}} = (1_{[n-v+1]}, \overline{a_{r+n+1}, \cdots a_p, a_0, a_1, \cdots a_r, 1_{[v-1]}})$

$= f$ (say) when $1 \leqslant v \leqslant n$ ... (2·1)

By Galois' theorem⁽⁴⁾ of inverse periods,

$$\frac{Q_{r+v}}{\sqrt{R}-P_{r+v}} = (1_{[v-1]}, a_r, \cdots a_1, \overline{a_0, a_p, \cdots a_{r+n+1}, 1_{[n-v+1]}})$$

i.e. $\dfrac{\sqrt{R}-P_{r+v}}{Q_{r+v}} = (0, \overline{1_{[v-1]}, a_r, \cdots 1_{[n-v+1]}}) = f'$ (say) ... (2·2)

Adding (2·1) and (2·2), $\dfrac{2\sqrt{R}}{Q_{r+v}} = f+f'$, so that Q_{r+v} is greater or less than $\sqrt{R}$ according as f is less or greater than $(2-f')$ or f'' (say).

Now, $2-f' = 2-(0, 1_{[v-1]}, \cdots)$
$=(1, 1, 0, 1_{[v-2]}, \cdots)$
$=(1, 2, 1_{[v-3]}, \cdots)$

If $v \geqslant 3$ and $(n-v+1) \geqslant 2$, i.e. $n-1 \geqslant v \geqslant 3$, the second complete quotient of f, is less than the corresponding quotient of f'' and therefore $f > f''$ so that $Q_{r+v} < \sqrt{R}$.

If $v=2$, and $n \geqslant 3$, we have $f''=(1, 1+a_r, \cdots) < f$ and again $Q_{r+v} < \sqrt{R}$.

Thus, for all values of v greater than 1 and less than $n (\geqslant 3)$, $Q_{r+v} < \sqrt{R}$.

The theorem is therefore proved.

Cor. (1) When $v = 1$, $n > 2$ and $a_r > 2$, we have $f < f'$ and therefore $Q_{r+1} > \sqrt{R}$.

If $v = 1$, $n = 2$, we find Q_{r+1} greater or less than $\sqrt{R}$ according as a_{r+3} is less or greater than $a_r - 1$; if $a_r = 2$, $Q_{r+1} < \sqrt{R}$.

If $n = v = 1$, $f < f''$ and $Q_{r+1} > \sqrt{R}$.

Cor. (2) When $v = n > 2$, we can show that $Q_{r+n} > \sqrt{R}$, if $a_{r+n+1} > 2$.

When $v = n = 2$, Q_{r+2} is greater or less than $\sqrt{R}$ according as a_{r+3} is greater or less than $a_r + 1$: if $a_{r+3} = 2$, $Q_{r+2} < \sqrt{R}$.

If $n = 2$ and $a_{r+3} = a_r$, both Q_{r+1} and Q_{r+2} are less than $\sqrt{R}$.

N. B. The complete quotients corresponding to the first and the last of a set of unit partial quotients preceded and followed by partial quotients greater than 1, may have their denominators greater or less than $\sqrt{R}$ according to circumstances and are thus distinguished from the other complete quotients whose denominators are always less than $\sqrt{R}$. From the[5] theory of continued fractions to the nearest square (or otherwise) it will be seen that in a pure recurring simple continued fraction equivalent to $\frac{P_0 + \sqrt{R}}{Q_0}$, the denominators of the complete quotients are all less than $\sqrt{R}$, when the corresponding partial quotients are greater than 1.

3. THEOREM II:—*In the S. C. F development of a standard surd of the form* $\frac{q + \sqrt{p^2 + q^2}}{p}$, $(p > 2q > 0)$ *there cannot occur a complete quotient of the same form more than once, in the recurring period; when such a complete quotient does occur, the recurring period is symmetric with an even number of terms, which include in the middle an even number of unit partial quotients.*

Proof:—Let $\xi_0 = \frac{P_0 + \sqrt{R}}{Q_0} = \frac{q + \sqrt{p^2 + q^2}}{p}$, $[p > 2q > 0.]$

Then it is readily seen that $1 < \xi_0 < \frac{\sqrt{5} + 1}{2} = (1\infty)$ ... (3·1)

while $\bar{\xi}_0$ (the conjugate of ξ_0) $= \frac{q - \sqrt{p^2 + q^2}}{p}$, so that

$$\xi_0 \bar{\xi}_0 = -1 \text{ and } -1 < \bar{\xi}_0 < 0.$$

Hence, by a well known theorem[6] of Galois, the S. C. F. development of ξ_0 has a pure recurring period, say $(\overline{a_0, a_1, \cdots a_n})$.

From (1), $a_0 = 1$ and if a_m be the first partial quotient greater than 1, m must be odd; for, if m be even, we have successively:

$$
\begin{aligned}
&(a_m, \cdots\cdots) > (1\infty),\\
&(1,\ a_m, \cdots\cdots) < (1\infty),\\
&(1[2],\ a_m, \cdots\cdots) > (1\infty),\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
&(1[m],\ a_m, \cdots) > (1\infty), \text{ which contradicts } (3{\cdot}1)
\end{aligned}
$$

Hence $$\xi_0 = \left(1[m],\ \overline{a_m, \cdots a_n}\right). \qquad \ldots \quad (3{\cdot}2)$$

Again, from the theorem of inverse periods,

$$\xi_0 = -\frac{1}{\bar{\xi}_0} = \left(a_n,\ \overline{\cdots 1[m]}\right) \qquad \ldots \quad (3{\cdot}3)$$

Comparing (3·2) and (3·3), we have

$$a_n = a_{n-1} = \cdots = a_{n-m+1} = 1,\ a_m = a_{n-m};$$

and the period is a symmetric one beginning and ending with an odd number of unit partial quotients.

The comparison of the complete quotients in (3·2) and (3·3) gives, when $v \leqslant n$,

$$\frac{P_v + \sqrt{R}}{Q_v} = \frac{P_{n+1-v} + \sqrt{R}}{Q_{n-v}}, \text{ so that } P_v = P_{n-v+1},\ Q_v = Q_{n-v} \ \ldots \quad (3{\cdot}4)$$

If $Q_v = Q_{v-1}$, then $\xi_v = \dfrac{P_v + \sqrt{R}}{Q_v} = \dfrac{P_{n-v+1} + \sqrt{R}}{Q_{v-1}} = \xi_{n-v+1}$

since $Q_{v-1} = Q_{n-v+1}$ by (3·4).

Hence, $v = n + 1 - v$, i.e. $v = \dfrac{n+1}{2}$.

This shows that, if two consecutive Q's should be equal, n must be odd and the equal Q's are $Q_{\frac{n+1}{2}}$ and $Q_{\frac{n-1}{2}}$ and no others. ... (3·5)

If a complete quotient, say ξ_v, of the recurring period of ξ_0 be of the same form as ξ_0, its S. C. F development should possess the same symmetries.

Writing $Q_0, Q_1, \cdots Q_n$ round a circle at the vertices of a regular polygon of $(n+1)$ sides, we find that they arrange themselves symmetrically about a diameter such that the Q's symmetrically placed about this diameter are also equal, since $Q_v = Q_{n-v}$.

The symmetry of the Q's corresponding to ξ_v imply that $Q_v = Q_{v-1}$ just as $Q_0 = Q_n$.

From (3·5), we see that there cannot be more than one ξ_v of the same form as ξ_0 within the recurring period and it can occur only when n is odd and $v = \frac{1}{2}(n+1)$.

Further, when n is odd, we realise the same symmetry of Q's starting from $Q_{\frac{n+1}{2}}$, going round the circle and ending with $Q_{\frac{n-1}{2}}$ as in the first set $(Q_0, Q_1, \cdots Q_n)$.

This proves the existence of ξ_v of the same form as ξ_0, only when $a_{(n \pm r)/2} = 1$, where $r = 1, 3, 5, \cdots (2m-1)$, and m, n are both odd.

Hence, if ξ_0 should have a remote successor of the same form as itself in the recurring period of its S. C. F development,

then $\xi_0 = (\overline{1_{[2k+1]}, a_{2k+1}, \cdots a_l, 1_{[4k'+2]}, a_l, \cdots a_{2k+1}, 1_{[2k+1]}})$
with a symmetric set of partial quotients, where a_{2k+1}, a_l are not equal to 1.

In order that the recurring cycle may preserve its character as a primitive period, it is necessary that

$a_{2k+1} \neq a_l$, $k \neq k'$, and $a_p \neq a_q$ where $(2k+1) < (p, q) < l$, the number of terms in the cycle being $2p + 2q + 2$.

Example: $\dfrac{27 + \sqrt{27^2 + 82^2}}{82} = (\overline{1, 2, 1_{[6]}, 2, 1})$ has a remote successor

$$\frac{37 + \sqrt{37^2 + 78^2}}{78} = (\overline{1_{[3]}, 2, 1_{[2]}, 2, 1_{[3]}})$$

4. We will next proceed to examine, under what circumstances a standard quadratic surd of the form $\sqrt{R}/Q_0$ can have a complete quotient of the form in Theorem II in its S. C. F development, say

$$(a_0, \overline{a_1, a_2, \cdots a_{k-1}, 2a_0}) \quad \ldots \quad (A)$$

Since the first and the last partial quotients in the period of $\dfrac{q + \sqrt{p^2 + q^2}}{p}$, $(p > 2q > 0)$, are unity, while $2a_0$ can never be equal to 1, a complete quotient, say ξ_v, if any, of the form in question occurring in the recurring period of (A), cannot correspond to either a_1 or $2a_0$, so that

$$\xi_v = (\overline{a_v, \cdots a_{v-1}}), \ a_v \neq a_1 \text{ or } 2a_0.$$

From the known property of the Q's, we have $Q_v = Q_{v-1}$, which implies by a well-known theorem[6] of Muir, that k must be odd, and in this case $\xi_v = \frac{P_v + \sqrt{R}}{Q_v}$, where $R = P_v^2 + Q_v^2$.

Further, there cannot be another complete quotient of the same form in the recurring period, which is possible only when k is even (vide (3·5) supra).

From the symmetries in the recurring period in (A), we infer that

$$\xi_{\frac{k+1}{2}} = (\overline{a_{\frac{k+1}{2}}, \cdot\cdot\, a_{k-1}, 2a_0, a_1, \cdot\cdot\, a_{\frac{k-1}{2}}}).$$

Thus $\sqrt{R}/Q_0$ must contain in its S. C. F period an even number $(4n+2)$ of unit partial quotients in the centre of the symmetric part of the period in order that one of its complete quotients may be of the form $\frac{q+\sqrt{p^2+q^2}}{p}$ $(p>2q>0)$.

Examples: $\sqrt{58} = (7, \overline{1[6], 14})$; $\sqrt{97} = (9, \overline{1, 5, 1[6], 5, 1, 18})$
We condense the above results in the following

THEOREM III:—*If the standard surd of the form* $(\sqrt{R})\, Q_0$ *have in its S. C. F development a complete quotient of the form* $\frac{q+\sqrt{p^2+q^2}}{p}$, $p>2q>0$, *then the symmetric portion of the recurring period of partial quotients will contain in the middle an even number (of the form* $4n+2$*) of unit partial quotients; and there cannot occur any other complete quotient of the same form within the recurring period.*

5. Before concluding, we wish to point out an application of the above theorems in connection with the most rapidly convergent half regular continued fractions.

According to[7] Tietze's theorem, the most rapidly convergent *h. r. c.* F's are characterised by the property that the complete quotients are everywhere greater than $\frac{1}{2}(\sqrt{5}+1)$. We have proved elsewhere that the only transformations (apart from the P-transformation) that go to convert a S. C. F into one of the most rapidly convergent *h. r.* C. F's are the annihilatory transformations which we have called the C_1, C_2, and $c_1\ c_2^e$ types. The effect of an annihilatory transformation applied to a unit partial quotient is obviously to increase the following complete quotient by 1, without affecting the preceding complete quotient.

From the above considerations, we see that a complete quotient of the form $\frac{q+p+\sqrt{p^2+q^2}}{p}$ will occur in one of the most rapidly convergent *h. r.* C. F developments of $(\sqrt{R})/Q_0$ (not involving a P-transformation), when and only when either $\frac{q+\sqrt{p^2+q^2}}{p}$ or $\frac{q+p+\sqrt{p^2+q^2}}{p}$ occurs in the corresponding S. C. F development.

Since $\frac{q+p+\sqrt{p^2+q^2}}{p}$ is not a reduced surd in(3) Perron's sense and cannot therefore occur in the S. C. F recurring period, while $\frac{q+\sqrt{p^2+q^2}}{p}$ can occur just once in the same period under the conditions of Theorem III, it follows that a most rapidly convergent *h. r.* C. F development of $(\sqrt{R})/Q_0$ (not involving a P-transformation) will have in its period $\frac{q+p+\sqrt{p^2+q^2}}{p}$ as a complete quotient just once, when the unit partial quotient corresponding to $\frac{q+\sqrt{p^2+q^2}}{p}$ in the S. C. F is not annihilated.

Let $(\sqrt{R})/Q_0 = (a_0, \overline{a_1, \cdots a_p, 1_{[4t+2]}, a_p, \cdots a_1, 2a_0})$, where ξ_{p+2t+2} is the only complete quotient of the form $\frac{q+\sqrt{p^2+q^2}}{p}$. The result of applying the C_1-transformation gives the complete quotient $1+\xi_{p+2t+2}$; if the C_2-transformation be applied, the unit partial quotient corresponding to ξ_{p+2t+2} will be annihilated and so there will be no complete quotient of the form in question. We have, therefore,

THEOREM IV:—*In the development of* $(\sqrt{R})/Q_0$ *as a continued fraction to the nearest integer, there cannot occur more than one complete quotient of the form* $\frac{p+q+\sqrt{p^2+q^2}}{p}$ *in the recurring period, while in the singular continued fraction development, such a complete quotient can never occur.*

To preserve the complete quotient $1+\xi_{p+2t+2}$, we may also apply the eclectic transformation $c_1\ c_2^t$, provided the C_1-process is continued at least until it annihilates the $(2t+1)$-th unit partial quotient.

Lastly, we may state that the C_1-transformation applied to odd batches of unit partial quotients and the $c_1\ c_2^t$ transformation to even batches of a pure recurring S. C. F, (ξ_0), will remove the special unit partial quotients and then, by Theorem I, the denominators of all complete quotients are necessarily less than $\sqrt{R}$.

The results set down in this paper are useful in the theory of the continued fraction to the nearest square.(8)

References

(1) A New Study of the Half-regular Continued Fraction, by A. A. Krishnaswami Ayyangar: (*The Mathematics Student*, Vol. 6. No. 2, pp. 45–67, 1938).

(2) ber die Annäherung an eine reele Grösse durch rationale Zahlen, by H. Minkowski. (*Mathematische Annalen*, Vol 54, 1901).

(3) Die Lehre von den Kettenbrüchen, by Oskar Perron (pp. 82, 85, 183, 184), 1929 (Leipzig and Berlin).

(4) Demonstration d'un theoreme sur les fractions continues periodiques by E. Galois. (*Annales de mathematiques pures et appliquées*, Vol. 19.)

(5) A New Continued Fraction, by A. A. Krishnaswami Ayyangar. (*Current Science*, Vol. VI, No. 12, pp. 602-604, 1938.)

(6) The expression of a quadratic surd as a continued fraction by Th. Muir, (Glasgow, 1874).

(7) Über die raschesten Kettenbruchentwicklungen reeller Zahlen, by H. Tietze (*Monatshefte für Mathematik und Physik*, Bd. 24, 1913.)

(8) Theory of the nearest square continued fraction, by A. A. Krishnaswami Ayyangar. (*Journal of the Mysore University*, Vol I, No. 2† Section A, New Series).

---

† This issue will be published shortly.

GLEANINGS

The same channels and the same relations which led the Arabs to a knowledge of Indian Algebra, enabled them also to obtain, in the ninth century, Indian numerals from Persia and the shores of Euphrates. Persians were established at that time as Revenue Collectors on the Indus, and the use of Indian numerals was gradually transmitted to the Revenue Officers of the Arabs in Northern Africa, opposite the shores of Sicily. Nevertheless, the important historical investigations of the distinguished mathematician Charles have rendered it more than probable according to his correct interpretation of the so-called Pythagorean table in the geometry of Bœthius, that the Christians in the west were familiar with Indian numerals even earlier than the Arabs, and that they were acquainted with the use of nine figures or characters according to their position value under the name of the system of the abacus.

HUMBOLDT,

Quoted from *Fabre and Mathematics and other essays*, p. 39

THE INFINITESIMAL FLOW IN THE INVERSIVE PLANE

BY

G. SRINIVASAN, *Research Student, Annamalai University.*

Given any continuous r-parameter group G_r of point transformations we can associate with it a system of r infinitesimal operators

$$X_r = \sum \xi_r \frac{\partial}{\partial x_r}$$

such that any infinitesimal transformation can be expressed in the form

$$x_i' = x_i + \sum_a c^a \xi_a^i \delta t$$

where δt is a scalar infinitesimal. Interpreting δt as a time interval these operators determine a velocity field **v** which is charateristic of this group. I study here the velocity field **v** due to the inversive group.

The Infinitesimal operators of the Inversive group:

Topologically the real inversive plane is homeomorphic with the 2 sphere, the correspondence between points being most easily established by streographic projection. Under such a correspondence the operators of the inversive group in the plane correspond to projective transformations which transforms the quadric into itself. To obtain the infinitesimal operators, we notice that $\lambda = x + iy$ and $\mu = x - iy$ may be taken as the "generator coordinates" on the sphere and correspond to the 2 sets of isotropic lines in the cartesian plane. A projective transformation of the sphere which does not interchange the 2 systems of generators correspond to the linear group $\lambda' = (a_1\lambda + b_1)/(c_1\lambda + d_1)$ on the one system together with a transformation $\mu' = (a_2\mu + b_2)/(c_2\mu + d_2)$ of μ generators. The infinitesimal operators in the λ and μ coordinates are thus:

$$\frac{\partial}{\partial\lambda}, \lambda \frac{\partial}{\partial\lambda}, \lambda^2 \frac{\partial}{\partial\lambda}, \frac{\partial}{\partial\mu}, \mu \frac{\partial}{\partial\mu}, \mu^2 \frac{\partial}{\partial\mu}. \qquad \dots (1)$$

From these we derive by substituting in terms of x and y on forming real combinations the infinitesimal operators of the inversive group of the plane, namely:

$$\text{(I)} \ \frac{\partial}{\partial x}, \qquad \text{(II)} \ \frac{\partial}{\partial y};$$

$$\text{(III)}\quad x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x};$$

$$\text{(IV)}\quad x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y};$$

$$\text{(V)}\quad 2xy\frac{\partial}{\partial x}+(y^2-x^2)\frac{\partial}{\partial y},$$

$$\text{(VI)}\quad 2xy\frac{\partial}{\partial y}+(x^2-y^2)\frac{\partial}{\partial x}. \qquad \dots \quad (2)$$

Of these, the first two operators represent small displacements parallel to the x and y axes respectively; the third represents a small rotation about the origin; the fourth represents a small uniform expansion with the origin as a fixed point. The first three of these generate the *metrical group*, while if the fourth is also adjoined we have the *affine group*. The projective group of the plane is obtained by adjoining to the operators, (I), (II), (III), and (IV) the operators

$$x\frac{\partial}{\partial y}+y\frac{\partial}{\partial x},\quad x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y},\quad x^2\frac{\partial}{\partial x}+xy\frac{\partial}{\partial y},\quad xy\frac{\partial}{\partial x}+y^2\frac{\partial}{\partial y}.$$

The operators V and VI are thus peculiar to the inversive group. The fifth corresponds to an inversion with respect to a circle of unit radius whose centre is at the origin followed by an inversion with respect to a circle of unit radius whose centre is at a small distance δt from the origin on the y axis followed by a translation back-wards along the axis of y measured by δt. The operator VI has a similar interpretation in relation to the axis of x.

Other subgroups that can be generated from these operators are:—

(*a*) The group generated by the operators III, IV, V and VI which keeps the origin invariant and (*b*) groups generated by the operators I, IV and VI and II, IV, V; the group generated by I, IV and VI keeps the x axis invariant and the other keeps the y axis invariant.

The Components of the Flow Vector. Any general infinitesimal operator X of the Inversive group of the plane G_6 is linearly dependent upon the above six operators. Hence it may be represented by

$$X\equiv a_0\frac{\partial}{\partial x}+a_1\frac{\partial}{\partial y}+a_2\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)+a_3\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)$$
$$+a_4\left\{2xy\frac{\partial}{\partial x}+(y^2-x^2)\frac{\partial}{\partial y}\right\}+a_5\left\{2xy\frac{\partial}{\partial y}+(x^2-y^2)\frac{\partial}{\partial x}\right\}$$

$$\equiv \left\{ a_0 - a_2\, y + a_3\, x + 2a_4\, xy + a_5\, (x^2 - y^2) \right\} \frac{\partial}{\partial x}$$

$$+ \left\{ a_1 + a_2\, x + a_3\, y + a_4\, (y^2 - x^2) + 2a_5\, xy \right\} \frac{\partial}{\partial y} \qquad \dots \quad (3)$$

where a_0, a_1, a_2, a_3, a_4, a_5, are constants.

The velocity vector **v** is given by

$$\mathbf{v} = \mathbf{i}\; v_0 + \mathbf{j}\; v_1 \qquad \dots \quad (4)$$

where $v_0 = \{a_0 - a_2\, y + a_3\, x + 2a_4\, xy + a_5\, (x^2 - y^2)\}$

$v_1 = \{a_1 + a_2\, x + a_3\, y + a_4\, (y^2 - x^2) + 2a_5\, xy\}.$

Calculating div **v** and curl **v** from (3) we have:—

$$\text{div } \mathbf{v} = \frac{\partial v_0}{\partial x} + \frac{\partial v_1}{\partial y}$$

$$= 2\ (a_3 + 2a_4\, y + 2a_5\, x) \qquad \dots \quad (5)$$

and

$$\text{curl } \mathbf{v} \equiv \left(\frac{\partial v_1}{\partial x} - \frac{\partial v_0}{\partial y} \right) \mathbf{k}.$$

$$= 2\ (a_2 - 2a_4\, x + 2\, a_5\, y)\ \mathbf{k}. \qquad \dots \quad (6)$$

where **k** is a unit vector normal to the plane.

From equations (5) and (6) we see that the loci of points at which div **v** and curl **v** are constants are given by two systems of lines which cut orthogonally.

Taking the lines along which div **v** and curl **v** vanish as the axes of x and y respectively we have

$$a_3 + 2a_4\, y + 2a_5\, x \equiv y' \equiv 0, \text{ i.e. } a_3 \equiv a_5 \equiv 0 \qquad \dots \quad (7)$$

and

$$a_2 - 2a_4\, x + 2a_5\, y \equiv x' \equiv 0, \text{ i.e. } a_2 \equiv a_5 \equiv 0 \qquad \dots \quad (8)$$

Hence by choosing this set of new axes we have the new velocity vector $\mathbf{v}'$ given by the equation $\mathbf{v}' = \mathbf{i}v_0' + \mathbf{j}v_1'$ ... (9)

where $v_0' = a_0 + 2a_4\, xy$

$v_1' = a_1 + a_4\, (x^2 - y^2).$

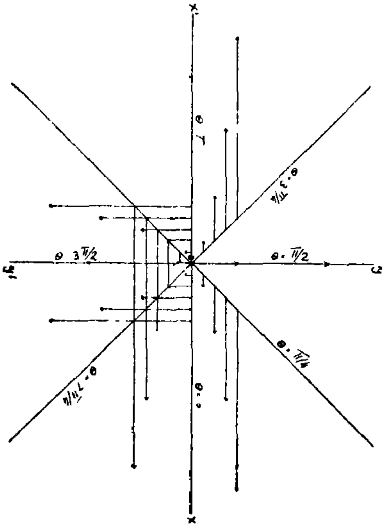
This velocity vector $\mathbf{v}'$ can be separated into two velocity vectors :—

(a) $\mathbf{v}_1 = \mathbf{i}a_0 + \mathbf{j}a_1$ due to a pure translation :

and (b) $\mathbf{v}_2 = 2a_4\, xy\, \mathbf{i} + a_4\, (y^2 - x^2)\, \mathbf{j}$.

which is peculiar to the inversive group.

The velocity vector corresponding to $\mathbf{v}_2$ is obtained as follows :— changing to polar coordinates we see that the displacements along the x and y axes are given by $r^2 \cos(2\theta - 90°)$ and $r^2 \sin(2\theta - 90°)$.



From the above formula it may be verified that the velocity vector at all points on a straight line through the origin are parallel and similarly directed while the magnitude varies as the square of the distance of the point from the origin. A fairly good idea of the direction and magnitude of the velocity vectors at different points in the plane, may be had from fig. 1.

The direction of the velocity vector can also be obtained as follows: Suppose the circle of unit radius with centre at O′ rolls without slipping on another unit circle with centre O when the point of contact is on the x axis (the line of centres being the x axis). Let AB be the diameter parallel to the y axis through O′. Then the direction of AB gives the direction of the velocity vector of O′.

Before concluding I wish to express my thanks to Professor A. Narasinga Rao for suggesting this problem and for his guidance in this investigation.

GLEANINGS

The geometry of your face,
The algebra of your eyes—with their knowns and unknowns,
The calculus of your personality—with its constants and variables,
The radii of your charm that encircles me,
Your very locus,
The force of your smile, that produced becomes the perpendicular bisector of my heart,
And leaves me entangled in the graph you have plotted around me :—

CATHERINE M. WILLIMEN,
(*From The Mathematics Teacher, Feb. 1941.*)

THE LINE OF UMBILICS

BY

DR. C. N. SRINIVASIENGAR, *Bangalore.*

1. At an umbilic on a surface, it is well-known that there are three principal directions in the sense that the normal at the umbilic meets the normal at a consecutive point. The object of this note is to point out the following result, which does not appear to have been noticed, as far as I am aware.

If a surface possesses a locus of umbilics, the tangent to this locus at any point is one of the three principal directions thereat. In other words, a line of umbilics on a surface is a line of curvature.

Taking the origin at an umbilic, the equation of the surface may be written in the form:

$$2z = \frac{x^2+y^2}{\rho} + \tfrac{1}{3}\,(ax^3+3bx^2y+3cxy^2+dy^3) + \cdot\cdot\cdot\cdot$$

If $(x, y, 0)$ be a neighbouring umbilic, x and y being small, we obtain from the differential equations

$$\frac{1+p^2}{r} = \frac{pq}{s} = \frac{1+q^2}{t},$$

$$\frac{1+\dfrac{x^2}{\rho^2}+\cdot\cdot\cdot}{\dfrac{1}{\rho}+ax+by+\cdot\cdot\cdot} = \frac{xy/\rho^2}{bx+cy+\cdot\cdot\cdot} = \frac{1+\dfrac{y^2}{\rho^2}}{\dfrac{1}{\rho}+cx+dy+\cdot\cdot\cdot}.$$

It follows that the denominator of the second fraction should be an infinitesimal of the second order. Hence, to the first order of infinitesimals, we must have

$$bx+cy=0$$

$$\text{and } (a-c)\,x+(b-d)\,y=0.$$

Hence, $$b\,(b-d)=c\,(a-c). \qquad (1)$$

Now, the three principal directions at O are given by*

$$c\tan^3\alpha+(2b-d)\tan^2\alpha-(2c-a)\tan\alpha-b=0 \qquad (2)$$

---

* Bell. *Coordinate Geometry of Three Dimensions,* p. 343.

where α is the angle that any of the directions makes with the x-axis. It is seen that tan $\alpha = y/x = -b/c$ satisfies the equation (2) in virtue of the relation (1). This proves the result.

2. From Rodrigue's formulae for a line of curvature, we have

$$\frac{dx}{dX} = \frac{dy}{dY} = \frac{dz}{dZ}$$

where X, Y, Z are the direction cosines of the normal at (x, y, z). It is now easily seen that these equations are satisfied also, by a line of umbilics if such a line be present on any surface. This fact has a bearing on the following theorem of Joachimsthal namely:

If two surfaces intersect at a constant angle, and if the curve of intersection is a line of curvature upon one of the surfaces, it is a line of curvature on the other also.

Two proofs* are known for this theorem, one by using Rodrigue's formulae, the other by using the formulae for geodesic torsion. The second method will explain that the curve of intersection can also possibly be a line of umbilics on the second surface. The property proved in §1 of this note serves to explain that Joachimsthal's theorem remains *literally* true even in this exceptional case. That the first method of proof also furnishes this alternative is now clear from the remark made above regarding Rodrigue's formulae.

---

* Weatherburn, *Differential Geometry* I, Arts. 29 and 49.
Forsyth, *Differential Geometry*, Arts. 107 and 128.

NOTES AND DISCUSSIONS

A Proof of an elementary theorem on complex numbers.

If two complex numbers w_1, w_2 are derived from two complex numbers z_1, z_2 by the pair of formulae

$$w_1 = az_1 + bz_2, \qquad w_2 = cz_1 + dz_2,$$

in which the coefficients a, b, c, d are real, then rotation through an angle less than two right angles from w_1 to w_2 is in the same direction as rotation through an angle less than two right angles from z_1 to z_2 or is in the opposite direction according as $ad - bc$ is positive or negative.

This theorem is familiar, but it is usually proved by dissection of the complex numbers into real and imaginary parts, and a proof in which there is no dissection is interesting.

The coefficients c, d are not both zero, and we suppose c not to be zero. Then writing Z, W for $z_1 z_2$, $w_1 w_2$ we have

$$\mathrm{W} = \frac{a\mathrm{Z} + b}{c\mathrm{Z} + d},$$

that is,

$$\left(\mathrm{W} - \frac{a}{c}\right)\left(\mathrm{Z} + \frac{d}{c}\right) = \frac{bc - ad}{c^2}.$$

Since the product of the two complex numbers

$$\mathrm{W} - \frac{a}{c}, \qquad \mathrm{Z} + \frac{d}{c}$$

is real, the first of these complex numbers is a real multiple of the conjugate of the second. If the multiplying factor is positive, then (i) the imaginary parts of the two complex numbers have opposite signs, (ii) the product of the two complex numbers is positive. On the other hand, if the multiplying factor is negative, then (i) the imaginary parts of the two complex numbers have the same sign, (ii) the product of the two complex numbers is negative. But (i) since a/c, d/c are real, the imaginary parts of the two complex factors on the left-hand side of the last equation are simply the imaginary parts of W and Z themselves; (ii) since c^2 is essentially positive, the product is positive or negative according as $ad - bc$ is negative or positive. Thus the imaginary parts of W and Z have opposite signs or the same sign according as $ad - bc$ is negative or positive, and this is the result which was to be established.

Sonning on Thames,
England.

E. H. NEVILLE.

Power Series for Circular Functions.

1. The juvenile enthusiast in Trigonometry who has mastered the elegant expansions in powers of x for $\sin x$ and $\cos x$ is apt to be disappointed when he is abruptly told, as in Loney's *Plane Trigonometry* (Part II), article 34, that 'there is no series, proceeding according to a simple law, for the expansion of $\tan x$ in powers of x, and though we can find the series for $\tan x$ to as many places as we please, the method however soon becomes very cumbrous and troublesome'. In what follows assuming the existence of the series in question, I obtain the coefficients in the power series for $\tan x$ and $\cot x$, $\operatorname{cosec} x$ and $\sec x$ in the form of elegant determinants.

2. Since $\tan x$ is an odd function of x we assume

$$\tan x = T_1 x + T_3 x^3 + T_5 x^5 + \cdots + T_{2k-1} x^{2k-1} + \cdots \qquad \ldots \quad (1)$$

Then the identity $\sin x = \cos x \times \tan x$ gives

$$\begin{aligned} &\left\{ x - \frac{x^3}{3!} + \cdots + (-1)^{k-1} \frac{x^{2k-1}}{(2k-1)!} + \cdots \right\} \\ &= \left\{ 1 - \frac{x^2}{2!} + \cdots + (-1)^k \frac{x^{2k}}{(2k)!} + \cdots \right\} \\ &\times \left\{ T_1 x + T_3 x^3 + \cdots + T_{2k-1} x^{2k-1} + \cdots \right\} \qquad \ldots \quad (2) \end{aligned}$$

whence, equating the coefficients of x^{2k-1} on both sides, we have

$$\frac{1}{(2k-1)!} - \frac{T_1}{(2k-2)!} + \frac{T_3}{(2k-4)!} - \cdots + (-1)^{k-1} \frac{T_{2k-3}}{2!} + (-1)^k T_{2k-1} = 0 \qquad (3)$$

Putting in (3) $k = 1, 2, 3, \cdots k$ in succession, we have

$$\begin{aligned} &\frac{1}{1!} - T_1 &&= 0 \\ &\frac{1}{3!} - \frac{T_1}{2!} + T_3 &&= 0 \\ &\frac{1}{5!} - \frac{T_1}{4!} + \frac{T_3}{2!} - T_5 &&= 0 \quad \ldots \quad (4) \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots && \\ &\frac{1}{(2k-1)!} - \frac{T_1}{(2k-2)!} + \frac{T_3}{(2k-4)!} \cdots + (-1)^{k-1} \frac{T_{2k-3}}{2!} + (-1)^k T_{2k-1} &&= 0 \end{aligned}$$

From these k equations eliminating the $(k-1)$ quantities

$-T_1,\ T_3,\ -T_5,\ \cdots\ (-1)^{k-1}\ T_{2k-3}$, we obtain

$$T_{2k-1}=\left|\begin{array}{cccccc} \frac{1}{1!} & 1 & 0 & 0 & \cdots & 0 \\ \frac{1}{3!} & \frac{1}{2!} & 1 & 0 & \cdots & 0 \\ \frac{1}{5!} & \frac{1}{4!} & \frac{1}{2!} & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \frac{1}{(2k-1)!} & \frac{1}{(2k-2)!} & \frac{1}{(2k-4)!} & \cdot & \cdots & \frac{1}{2!} \end{array}\right| = \Delta(k), \text{ say, } \ldots \quad (5)$$

Setting $k=1,\ 2,\ 3,\ \cdots$ in (5) it is easily calculated that

$T_1=1,\ T_3=\frac{1}{3},\ T_5=\frac{2}{15},\ T_7=\frac{17}{315},\ \cdots\cdot$ Thus we have

$$\tan x = x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17}{315}x^7+\cdots+\Delta(k)\,x^{2k-1}+\cdots\cdot \quad \ldots\ (6)$$

3. Similarly, we observe that $x\cot x$ is an even function, such that $\lim\limits_{x\to 0}(x\cot x)=1$. Hence we write

$$x\cot x = 1+C_2x^2+C_4x^4+\cdots+C_{2k}x^{2k}+\cdots\cdot \quad \ldots\ (7)$$

The identity $\cos x=\left(\frac{\sin x}{x}\right)(x\cot x)$ then gives on equating the coefficient of x^{2k} on both sides, after a slight reduction,

$$\frac{2k}{(2k+1)!}+\frac{C_2}{(2k-1)!}-\frac{C_4}{(2k-3)!}+\cdots+(-1)^k\frac{C_{2k-2}}{3!}+(-1)^{k-1}C_{2k}=0 \ldots \quad (8)$$

Putting in (9), $k=1,\ 2,\ 3,\ \ldots,\ k$ and eliminating the $(k-1)$ quantities $C_2, -C_4, \ldots (-1)^k C_{2k-2}$ from the k equations thus obtained, we find

$$C_{2k}=\left|\begin{array}{cccccc} \frac{2}{3!} & 1 & 0 & 0 & \cdots & 0 \\ \frac{4}{5!} & \frac{1}{3!} & 1 & 0 & \cdots & 0 \\ \frac{6}{7!} & \frac{1}{5!} & \frac{1}{3!} & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & 0 \\ \frac{2k}{(2k+1)!} & \frac{1}{(2k-1)!} & \frac{1}{(2k-3)!} & \cdot & \cdots & \frac{1}{3!} \end{array}\right| = \nabla(k), \text{ say} \ldots \quad (9)$$

Setting $k=1, 2, 3, \ldots.$ in (9) we obtain $C_2 = -\frac{1}{3}, C_4 = -\frac{1}{45}, \cdots$ so that

$$x \cot x = 1 - \frac{1}{3} x^2 - \frac{1}{45} x^4 - \cdots + \nabla(k)\, x^{2k} + \cdots \qquad \ldots \quad (10)$$

4. Taking cosec x, we observe x cosec x is an even function and that $\lim_{x \to 0} (x \operatorname{cosec} x) = 1$. Hence we write

$$x \operatorname{cosec} x = 1 + P_2\, x^2 + P_4\, x^4 + \cdots + P_{2k}\, x^{2k} + \cdots \qquad \ldots \quad (11)$$

Using identity $(x \operatorname{cosec} x)\left(\frac{\sin x}{x}\right) = 1$ and proceeding as in the previous cases, we get

$$P_{2k} = \begin{vmatrix} \frac{1}{3!} & 1 & 0 & 0 & \cdots & 0 \\ \frac{1}{5!} & \frac{1}{3!} & 1 & 0 & \cdots & 0 \\ \frac{1}{7!} & \frac{1}{5!} & \frac{1}{3!} & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \frac{1}{(2k+1)!} & \frac{1}{(2k-1)!} & \frac{1}{(2k-3)!} & \cdot & \cdots & \frac{1}{3!} \end{vmatrix} = D(k), \text{ say}, \ldots \quad (12)$$

Putting $k = 1, 2, 3, \ldots.$ we get $D_2 = \frac{1}{6}, D_4 = \frac{7}{360}, \ldots.$ so that

$$x \operatorname{cosec} x = 1 + \frac{1}{6} x^2 + \frac{7}{360} x^4 + \cdots + D(k)\, x^{2k} + \cdots \quad \ldots \quad (13)$$

The corresponding series for sec x is well-known. We assume

$$\sec x = 1 + E_2\, x + E_4\, x^4 + \cdots + E_{2k}\, x^{2k} + \cdots. \qquad .. \quad (14)$$

and from the identity $\cos x \times \sec x = 1$ it follows by using the same method, we obtain the well known expression for the n^{th} Eulerian number:

$$E_{2k} = \begin{vmatrix} \frac{1}{2!} & 1 & 0 & 0 & \cdots & 0 \\ \frac{1}{4!} & \frac{1}{2!} & 1 & 0 & \cdots & 0 \\ \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \frac{1}{(2k)!} & \frac{1}{(2k-2)!} & \frac{1}{(2k-4)!} & \cdot & \cdots & \frac{1}{2!} \end{vmatrix} \qquad \ldots \quad (15)$$

It is easily seen that $E_2 = 1, E_4 = 5, E_6 = 61, \cdots$ so that

$$\sec x = 1 + x^2 + 5x^4 + 61\, x^6 + \cdots + E_{2k}\, x^{2k} + \cdots \qquad \ldots \quad (16)$$

5. A comparison of the series (6), (10), and (13) we have obtained above with the expansions of the corresponding circular functions in terms of Bernoulli's Numbers, gives rise to certain expressions for Bernoulli's Numbers in the form of determinants. Thus, it is known ‡ that

$$\tan x = \sum_{1}^{\infty} (-1)^{k-1} 2^{2k} (2^{2k}-1) \frac{B_{2k}}{(2k)!} x^{2k-1} \qquad \ldots (17)$$

$$x \cot x = \sum_{0}^{\infty} (-1)^{k} 2^{2k} \frac{B_{2k}}{(2k)!} x^{2k} \qquad \ldots (18)$$

and since $x \operatorname{cosec} x = x \cot \frac{x}{2} - x \cot x$, we have,

$$x \operatorname{cosec} x = \sum_{0}^{\infty} (-1)^{k-1} (2^{2k}-2) \frac{B_{2k}}{(2k)!} x^{2k} \cdots \qquad \ldots (19)$$

Now, comparing (17) with (6), (18) with (10), and (19) with (13), we obtain

$$B_{2k} = (-1)^{k-1} \frac{(2k)!}{2^{2k}(2^{2k}-1)} \Delta(k) \qquad \ldots (20);$$

$$= (-1)^{k} \frac{(2k)!}{2^{2k}} \nabla(k) \qquad \ldots (21);$$

$$= (-1)^{k-1} \frac{(2k)!}{(2^{2k}-2)} D(k) \qquad \ldots (22);$$

where $\Delta(k)$, $\nabla(k)$ and $D(k)$ are determinants of order k, defined in (5), (9) and (12) respectively.

Incidentally, from the above, we note the following relations between these three determinants ;

$$\Delta(k) : \nabla(k) : D(k) = 2^{2k}(1-2^{2k}) : 2^{2k} : (2-2^{2k}) \qquad \ldots (23),$$

the constant of proportionality being $(-1)^{k} \frac{B_{2k}}{(2k)!}$.

University of Travancore. K. BALACHANDRAN.

An alternate method of finding out square root of a Number

1. In this note an alternative process for obtaining square root of a number is explained. The method, though easy as it requires no multiplication at all is not always short. Subtractions are carried on after proper adjustment of digits. It has another advantage that, in case of perfect squares, the root can easily be checked.

---

† Vide Milne-Thomson: *Calculus of Finite Differences*, p. 148.

‡ Vide Knopp : *Theory and Application of Infinite Series*, p. 417.

2. Let $S_n = a+(a+b)+(a+2b) \cdot \cdot \cdot \cdot \cdot \cdot + \{a+(n-1)\,b\}$

$$= na + \frac{n\,(n-1)}{2}\,b$$

$$= n^2 \text{ if } a=1 \text{ and } b=2$$

So we get, $n^2 - \{1+3+5 \cdot \cdot \cdot \cdot \cdot \cdot \cdot +(2n-1)\} = 0$,

or to obtain the square root of a number, the successive terms of the series 1, 3, 5. 7, ... are continually subtracted from the given number, the number of steps at which the remainder vanishes being the required root.

Check: The nth term should be $2n-1$, i.e. the nth term should be 1 less than the double of the root.

3. Let us take as an illustration, the evaluation of $\sqrt{(729)}$.

$$\begin{array}{ccc l}
7 & 2 & 9 & \\
1' & & & \\
\cline{1-3}
6 & & & \\
3 & & & \\
\cline{1-3}
 & & & \text{(Corresponds to 20 subtractions)} \\
3 & 2 & 9 & \\
 & 4 & 1 & \\
\cline{1-3}
2 & 8 & 8 & \\
 & 4 & 3 & \\
\cline{1-3}
2 & 4 & 5 & \\
 & 4 & 5 & \\
\cline{1-3}
2 & 0 & 0 & \\
 & 4 & 7 & \\
\cline{1-3}
1 & 5 & 3 & \\
 & 4 & 9 & \\
\cline{1-3}
1 & 0 & 4 & \\
 & 5 & 1 & \\
\cline{1-3}
 & 5 & 3 & \\
 & 5 & 3 & \\
\cline{1-3}
 & & & \text{The required value is 27.}
\end{array}$$

Ths first 2 steps are equivalent to subtracting $100+300=20^2$ $=1+3+5 \cdot \cdot \cdot$ (20 terms). The third step requires explanation.

As 5 cannot be subtracted from 3, 29 is carried after three. Now the 21st. (10+10+1) subtraction is reached. So the number to be subtracted is $2 \times 21 - 1$ or 41. To make the process mechanical, it may be stated that 4 which is 1 less than the omitted term 5 of the arithmetic progression is placed below the ten's place (2) and 1 again is placed below the unit's place (9), the process of subtraction being carried on with regular increments of 2. Such are the only steps which may seem a bit complicated at the first sight, but will become easy with a little practice.

To obtain the root, the steps are counted as follows. *Two* times the digits have been subtracted continually. Next shifting has been necessary and then *seven* more continuous subtractions, the reminder vanishes. So the root is 27.

Check: $53+1=2\times 27$, which checks the result.

2·2. As another example we shall find the square root of 54856666225.

```
5,4 8 5 6 6 6 6 2 2 5 ┐               1 0 0 6 6       ┐
1                     │                 4 6 8 1       │
-------------         │               -------------   │
4                     ├ 2 steps         5 3 8 5       ├ 2 steps
3                     │                 4 6 8 3       │
-------------         ┘               -------------   ┘
1 4 8                 ┐               7 0 2 6 2       ┐
  4 1                 │               4 6 8 4 1       ┘ 1 step
-------------         │               -------------
1 0 7                 │               2 3 4 2 1 2 5   ┐
  4 3                 ├ 3 steps         4 6 8 4 2 1   │
-------------         │               -------------   │
  6 4                 │               1 8 7 3 7 0 4   │
  4 5                 │                 4 6 8 4 2 3   │
-------------         ┘               -------------   │
1 9 5 6               ┐               1 4 0 5 2 8 1   ├ 5 steps
  4 6 1               │                 4 6 8 4 2 5   │
-------------         │               -------------   │
1 4 9 5               │                 9 3 6 8 5 6   │
  4 6 3               │                 4 6 8 4 2 7   │
-------------         ├ 4 steps       -------------   │
1 0 3 2               │                 4 6 8 4 2 9   │
  4 6 5               │                 4 6 8 4 2 9   ┘
-------------         │               -------------
  5 6 7               │
  4 6 7               │
-------------         ┘
```

Therefore the required square root is 234215.

Again, $468429+1=2\times 234215$, which shows that the root is correct.

3. A more complicated case where the given number is not a perfect square is worked out below.

Find the square root of 184·25347 upto three places of decimal.

```
1 8 4·2 5, 3 4, 7 0                    2 0 0 3 4
1                                        2 7 0 1
────────────        1 step             ────────────
8 4                                    1 7 3 3 3
2 1                                      2 7 0 3
──────                                 ────────────
6 3                                    1 4 6 3 0
2 3                                      2 7 0 5
──────                                 ────────────
4 0                                    1 1 9 2 5
2 5                                      2 7 0 7
────────────        3 steps            ────────────
1 5·2 5                                  9 2 1 8
  2 6 1                                  2 7 0 9
──────                                 ────────────
1 2 6 4                                  6 5 0 9
  2 6 3                                  2 7 1 1
──────                                 ────────────
1 0 0 1                                  3 7 9 8
  2 6 5                                   2 7 1 3
──────                                 ──────────────── 7 steps
  7 3 6                                  1 0 8 5 7 9
  2 6 7                                    2 7 1 4 1
──────                                 ────────────
  4 6 9                                  8 1 4 2 9
  2 6 9                                   2 7 1 4 3
────────────        5 steps            ────────────
                                          5 4 2 8 6
                                          2 7 1 4 5
                                       ────────────
                                          2 7 1 4 1
                                          2 7 1 4 7
                                       ──────────────── 4 steps
                                             etc.
```

Therefore the required root =13·574 (approximately).

N. BAGCHI,
Statistics and Information Section
Indian Jute Committee.
Calcutta.

Pedal Conic

Let P and Q be two points in the plane of a triangle ABC such that the lines drawn through Q parallel to PA, PB, PC meet BC, CA, AB respectively in three collinear points. We may call Q a *pedal point* of P with respect to the triangle.

Taking ABC as the triangle of reference let the areal co-ordinates of P, Q be $(x_1\ y_1\ z_1)$, $(x_2\ y_2\ z_2)$. Then, the condition that Q may be a pedal point of P is easily seen to be $\sum \frac{1}{x_2}\left(\frac{1}{y_1}+\frac{1}{z_1}\right)=0$

which is also the condition that P may be a pedal point of Q showing that the relation between P and Q is reciprocal. Also the locus of pedal points of P is the circumconic $\sum x_1 (y_1+z_1)\, yz=0$. This conic may be called the *pedal conic* of P and the point P the *pedal point* of the circumconic. Further it is easily seen that to every circumconic there is a unique pedal point and that if the pedal conic of P passes through Q the pedal conic of Q passes through P. As a particular case we note that the pedal conic of the orthocentre is the circumcircle and that the pedal conic of any point on the circumcircle is a rectangular hyperbola.

The coordinates of the centre O of the pedal conic of $(x_1\ y_1\ z_1)$ are $\left(\frac{y_1+z_1}{2}, \frac{z_1+x_1}{2}, \frac{x_1+y_1}{2}\right)$. Hence if G be the centroid of the triangle, then $GP : OG = 2 : 1$ from which we infer that if a point P traces any curve, the centre of its pedal conic traces a similar curve. We may also note that given four points, the pedal conic of each of them with respect to the triangle formed by the other three is similar to the conic which is the locus of centres of conics through the four points.

K. O. AUGUSTINE.

Remarks on the above Note

Let ABC be any triangle, ω the line at infinity and P any point in the plane of the triangle. The conics through ABCP meet ω in point pairs of an involution range whose double points, say, α, β are the points at infinity on the centre locus Γ of the pencil of conics. We define the conic π through $ABC\alpha\beta$ the *pedal conic* of P with respect to ABC and show that this definition (which seems to be more natural than that given in the note mentioned above) immediately leads to the verification of the various relations shown to exist between the point P and its pedal conic π.

Among the conics through ABCP are three line pairs BC, AP; CA, BP; AB, CP whose meets with ω say L, X; M, Y; N, Z respectively, belong to the involution range and, therefore, separate harmonically the double points α, β. Hence, if Q be any point on π QL, QX etc., are parallel to conjugate diameters of π. And since QL is parallel to BC, QX is parallel to the diameter conjugate to BC. Thus QX, QY, QZ are parallel to the diameters conjugate to BC, CA, AB and, therefore,* meet BC, CA, AB respectively, in three collinear points.

---

* Taylor: *Geometrical conics* p. 366. The result is easily seen to be true by projecting π orthogonally into a circle.

Let Q be any point on π and Ψ its pedal conic. By the definition of the pedal conic the points at infinity on Ψ are harmonic with respect to α, β. Thus Ψ is a conic through ABCP and we have, therefore, the reciprocity relation that if the pedal conic of P passes through Q, then the pedal conic of Q passes through P. We may also note that the pedal conic of the orthocentre of the triangle is the circumcircle, follows immediately from the definition.

The other results mentioned in the note could be obtained very easily by projecting π orthogonally into a circle and using the well known properties of the circumcentre, orthocentre, centroid etc.

Annamalainagar K. RANGASWAMI.

A fallacy in deducing the attraction of a plane from that of a spherical shell

1. Whenever we get a result relating to a sphere of radius r, either in pure or in applied mathematics, it is not unusual, unless there is something to excite suspicion, to pass to the limit when $r \to \infty$ and thus obtain the corresponding result for a plane. Ordinarily we expect the theorem thus obtained to be correct. The example given below shows that the trouble may be much deeper than many of us are likely to suspect (unless one is of a highly suspicious temperament, which is by no means bad for his mathematics!) and should thus serve as a warning for the unwary.

Let O be the centre of a sphere of radius r and P a point distant $r+d$ from O, so that the nearest point Q of the sphere from P is at a distance d from it. When $r \to \infty$, the sphere becomes a plane distant d from P, and Q is the foot of the perpendicular from P on the plane.

The attraction at P of the spherical shell supposed to be of uniform surface density ρ is $\gamma 4\pi r^2 \rho/(r+d)^2$. When $r \to \infty$, this tends to the limit $4\pi\gamma\rho$, whereas it is known that the attraction of an infinite plane of density ρ at a point not on it is $2\pi\gamma\rho$.

The explanation of the discrepancy is quite simple. The sphere can be divided into 2 parts by the polar plane of P which exert equal attractions at P. When $r \to \infty$, the nearer segment which exerts half the attraction expands into the plane in question and hence the attraction of an infinite plane is only half of the limiting value obtained. The other segment becomes the plane at infinity and this exerts also an attraction $2\pi\gamma\rho$ at P. In other words, the limit of the sphere is a pair of planes, as projective geometry teaches us, one of these being the plane at ∞.

2. However, the projective group has no intrinsic relation to the problem and the paradox is one which should be traced to the inverse square law, or if we prefer the physics of continuity, its differential analogue the equation of Laplace. Taking the law of attraction to be the inverse nth power of the distance. we shall find that the integral giving the attraction of the sphere converges as $r \rightarrow \infty$ only for $n>2$. Dividing the spherical surface into two segments by the polar plane cf P, we shall find that for $n>2$, the attraction of the farther segment tends to zero with increasing r. The nearer segment expands out into infinite plane and its attraction becomes in the limit the attraction of the plane. Thus *in all cases except for $n=2$, no fallacy arises in this problem by replacing a sphere of infinite radius by the plane which is its limit*, and we get the correct value $2\pi\gamma\rho\,(n-1)d^{n-2}$. When $n=2$, a term of order r^{2-n} which represents the attraction of the farther segment tends to a finite limit instead of zero giving a wrong value to the attraction of the plane. If we prefer to put it that way, we may say that for $n>2$ the attraction of the plane at infinity is negligible while for $n=2$ it has got to be taken into account.

Annamalainagar. A. NARASINGA RAO.

GLEANINGS

NUPTIAL MATHEMATICS

One night, in a month of the spring season, a certain young lady.... was lovingly happy with her husband on the floor of a big mansion, white like the moon, and situated in a pleasure garden with trees bent down with the load of the bunches of flowers and fruits, and reasonant with the sweet sounds of parrots, cuckoos and bees which were all intoxicated with the honey obtained from the flowers therein. Then on a love quarrel arising between the husband and the wife, that lady's necklace, made up of pearls, became sundered and fell on the floor. One third of that necklace of pearls, reached the maidservant there; $\frac{1}{6}$ fell on the bed; then $\frac{1}{2}$ of what remained and $\frac{1}{2}$ of what remained thereafter, and again $\frac{1}{2}$ of what remained thereafter, and so on counting 6 times in all, fell all of them everywhere; and there were found to remain (unscattered) 1,161 pearls. If you know (how to work) miscellaneous problems (on fractions), give out the numerical measure of the pearls (in that necklace).

MAHAVIRACHARYA in *Ganitasarasangraha*
trans. by Prof. M. RANGACHARYA.

SOLUTIONS TO QUESTIONS

Question 1785

(V. THEBAULT, Le Mans, France):—Find three numbers A, B, C each of 4 digits such that the 16 digit number AABB may be the square of CC.

(Example: $1976\ 1976\ 8025\ 8025 = 4445\ 4445^2$)

Solution by G. Venkatasubbiah.

$CC = C \times 10001$. So $(CC)^2 = C^2 \times 10001^2$, i.e. $C^2 \times 100020001$. The multiplication may be set down as follows:—

Let $C^2 = abcdefgh$.

Multiplicand ...									a	b	c	d	e	f	g	h
Multiplier ...								1	0	0	0	2	0	0	0	1
									a	b	c	d	e	f	g	h
					$2a$	$2b$	$2c$	$2d$	$2e$	$2f$	$2g$	$2h$				
	a	b	c	d	e	f	g	h								
	A				A				B				B			

This multiplication will be correct only if $abcd + efgh = 10000$. So $abcd + efgh - 1 = 9999$. Hence $abcdefgh - 1$ i.e; $C^2 - 1$ is divisible by 9999. Now $9999 = 9 \times 11 \times 101$. Because $C+1$ and $C-1$ can have no common factor other than 2, the numbers 101, 11 and 9 must be the factors of either $C-1$ or $C+1$.

Let one of the two numbers $C-1$ and $C+1$ be $pqrs$; and let it be divisible by 101. Then $pq = rs$; i.e. the number is of the form $pqpq$. If the same number is divisible by 11, pq must be divisible by 11. If it is divisible by 9, pq is divisible by 9. But 9 or 11 might be the factor of the other number. The other number is $pqpq \pm 2$. If this must be divisible by 11 or 9, $pqpq \pm 101$ must be divisible by 9 or 11; $pq \pm 1$ must be divisible by 9 or 11.

Because $C \pm 1$ is $pqpq$, C must be $pqpq \pm 1$, where either pq or $pq \pm 1$ is divisible by 11 or 9.

The different values of C satisfying these conditions are 9998, 9899, 5554, 5455, 4544 and 4445.

Hence we have the following results:—

(i) $9997\ 9997\ 0004\ 0004 = 9998\ 9998^2$
(ii) $9800\ 9800\ 0201\ 0201 = 9899\ 9899^2$
(iii) $3085\ 3085\ 6916\ 6916 = 5554\ 5554^2$
(iv) $2976\ 2976\ 7025\ 7025 = 5455\ 5455^2$
(v) $2065\ 2065\ 7936\ 7936 = 4544\ 4544^2$
(vi) $1976\ 1976\ 8025\ 8025 = 4445\ 4445^2$

Also solved by P. A. Hariharan who adopts more or less the same method and shows that the same procedure can be followed when A, B, C are each of two or three digits. Also solved by S. Venkatanarasimhan.

Question 1788

(A. E. PALKAR):—Show that $2^{9961452}$ leaves the remainder 1374384229185 when divided by $(524287)^2$.

Solution by J. C. Auluck

If x is an odd number

$$(1+x)^x - 1 - x^2 \equiv 0 \ (\text{mod. } 2x^2)$$

because $(1+x)^x - 1 - x^2$ is even and is equal to

$$\frac{x\,(x-1)}{2\,!}\,x^2 + \frac{x\,(x-1)(x-2)}{3\,!}\,x^3 + \cdots + x^x \equiv 0 \ (\text{mod } x^2)$$

$$\text{Hence } \frac{(1+x)^x}{2} = \text{M}\,(x^2) + \frac{1+x^2}{2}$$

Thus $\frac{1}{2}(1+x)^x$ leaves the remainder $\frac{1}{2}(1+x^2)$ when divided by x^2.

The result follows on putting $x = 2^{19} - 1 = 524287$, the 19th Mersenne number, which is odd.

Also solved by Messrs. P. Hariharan, S. Venkatanarasimhan, G. Srinivasan, Venkataraman and C. R. Krishnamoorthi.

Question 1791

(P. KESAVA MENON):—If through any point O three secants are drawn to meet a conic in P,P′; Q,Q′; R,R′ respectively, then

$$\frac{\Delta \text{PQR}}{\text{OP}\cdot\text{OQ}\cdot\text{OR}} + \frac{\Delta \text{P}'\text{Q}'\text{R}'}{\text{OP}'\cdot\text{OQ}'\cdot\text{OR}'} = 0.$$

Solution by K. Balachandran.

Taking O as the origin, let the equation of the conic be

$$ax^2+2hxy+by^2+2gx+2fy+c=0. \qquad \ldots \quad (1)$$

Let OP, OQ, OR make angles θ_1, θ_2, θ_3 respectively with the x-axis.

Then it is obvious that OP, OP′ are the roots of the equation $r^2\,(a\cos 2\theta_1+2h\cos\theta_1\sin\theta_1+b\sin^2\theta_1)+2r\,(g\cos\theta_1+h\sin\theta_1)+c=0$

Hence we have

$$\frac{\mathrm{OP}+\mathrm{OP'}}{\mathrm{OP}\cdot\mathrm{OP'}}=-\frac{2\,(g\cos\theta_1+h\sin\theta_1)}{c} \qquad \ldots \quad (2)$$

and similar equations exist for $\dfrac{\mathrm{OQ}+\mathrm{OQ'}}{\mathrm{OQ}\cdot\mathrm{OQ'}}$ and $\dfrac{\mathrm{OR}+\mathrm{OR'}}{\mathrm{OR}\cdot\mathrm{OR'}}$.

Now

$$\begin{aligned}\Delta\mathrm{PQR}&=\Delta\mathrm{PQO}+\Delta\mathrm{QRO}-\Delta\mathrm{PRO}\\&=\tfrac{1}{2}\,\mathrm{OP}\cdot\mathrm{OQ}\sin(\theta_2-\theta_1)+\tfrac{1}{2}\,\mathrm{OQ}\cdot\mathrm{OR}\sin(\theta_3-\theta_2)\\&\qquad-\tfrac{1}{2}\,\mathrm{OP}\cdot\mathrm{OR}\sin(\theta_3-\theta_1)\end{aligned}$$

So that

$$\frac{2\Delta\mathrm{PQR}}{\mathrm{OP}\cdot\mathrm{OQ}\cdot\mathrm{OR}}=\frac{\sin(\theta_3-\theta_2)}{\mathrm{OP}}+\frac{\sin(\theta_1-\theta_3)}{\mathrm{OQ}}+\frac{\sin(\theta_2-\theta_1)}{\mathrm{OR}} \qquad \ldots \quad (3)$$

Similarly $$\frac{2\Delta\mathrm{P'Q'R'}}{\mathrm{OP'}\cdot\mathrm{OQ'}\cdot\mathrm{OR'}}=\frac{\sin(\theta_3-\theta_2)}{\mathrm{OP'}}+\frac{\sin(\theta_1-\theta_3)}{\mathrm{OQ'}}+\frac{\sin(\theta_2-\theta_1)}{\mathrm{OR'}} \qquad \ldots \quad (4)$$

Hence, by addition of (3) and (4),

$$\begin{aligned}2\left[\frac{\Delta\mathrm{PQR}}{\mathrm{OP}\cdot\mathrm{OQ}\cdot\mathrm{OR}}+\frac{\Delta\mathrm{P'Q'R'}}{\mathrm{OP'}\cdot\mathrm{OQ'}\cdot\mathrm{OR'}}\right]&=\sum\sin(\theta_3-\theta_2)\left(\frac{\mathrm{OP}+\mathrm{OP'}}{\mathrm{OP}\cdot\mathrm{OP'}}\right)\\&=-\frac{2}{c}\left[g\,\{\textstyle\sum\cos\theta_1\sin(\theta_3-\theta_2)\}+f\,\{\textstyle\sum\sin\theta_1\sin(\theta_3-\theta_2)\}\right]\end{aligned}$$

using (2)

$$=0.$$

Hence the result.

Also solved by Messrs. P. Hariharan, P. S. J. Manickavasagam, J. C. Auluck, G. Srinivasan and C. Radhakrishna Rao.

QUESTIONS FOR SOLUTION

1798. (K. BALACHANDRAN):

If $$\Delta(n, r) = \begin{vmatrix} \sigma_1 & 1 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ \sigma_2 & \sigma_1 & 2 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ \sigma_3 & \sigma_2 & \sigma_1 & 3 & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \sigma_{r-1} & \sigma_{r-2} & \sigma_r & & \cdot & \cdot & \cdot & \cdot & (r-1) \\ \sigma_r & \sigma_{r-1} & \sigma_{r-2} & \cdot & \cdot & \cdot & \cdot & \cdot & \sigma_1 \end{vmatrix},$$

where $\sigma_r = 1^r + 2^r + 3^r + \cdots + n^r$, show that

(i) $\Delta(n, n)$ is a perfect square.

(ii) $\Delta(n, n+j) \equiv 0$; $(j = 1, 2, 3, \ldots)$.

1799. (K. G. RAMANATHAN).—Prove that if l and m are positive integers and $l \geqslant m$, then

$$(3)^m \times (3)^l = (1)^{m-1}\ 0\ (9)^{l\ m}\ (8)^{m-1}\ 9$$

Deduce that if $a \leqslant 9$, then

$$(a)^l\ (0)^m - (a)^m\ (0)^l = (a)^{l-m}\ (0)^m$$

Here $(p)^l$ means a number formed by $ppp \cdot \cdot$ repeated l times.

1800. (A. A. KRISHNASWAMI IYENGAR): Show that

$$\int_0^x e^{-x^2/2}\,dx > \surd\left(\frac{\pi}{2}\right)\left(1 - \tfrac{1}{2}e^{-x^2} - \tfrac{1}{2}e^{-x^2/2}\right),$$

for all real x.

1801. (P. KESAVA MENON):—If

$$f(n) = \cos^{-1} 2^{-n/2}\,({}^n\mathrm{C}_0 - {}^n\mathrm{C}_2 + {}^n\mathrm{C}_4 - \cdots),$$

show that

$$f(p) + f(q) = f(p+q)$$